Dynamics of Arithmetic
A Connectionist View of Arithmetic Skills

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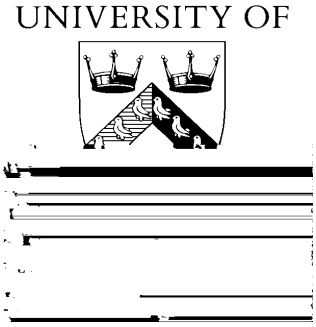
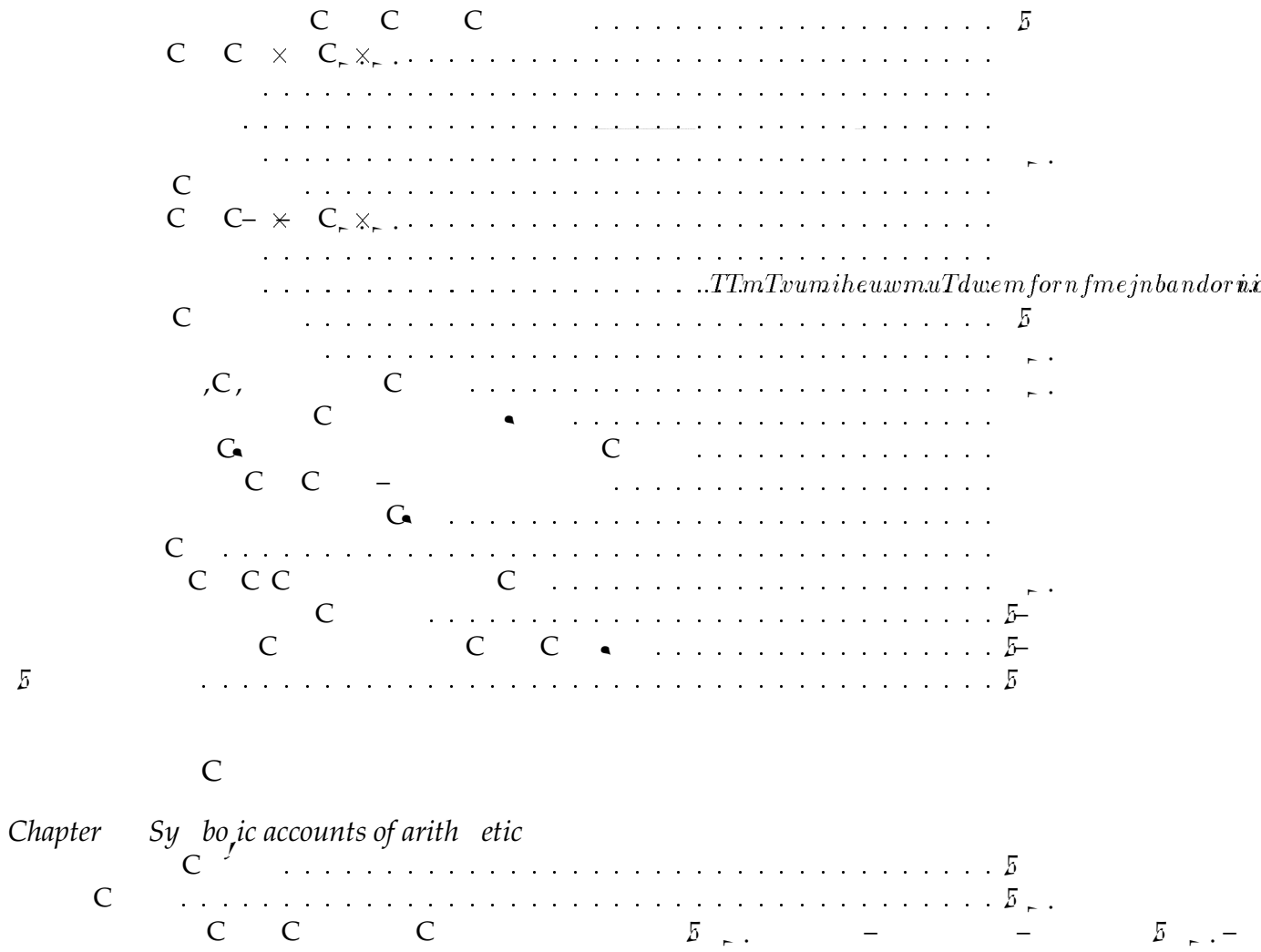
CSRP 306

February 1994

ISSN 1350-3162



Contents



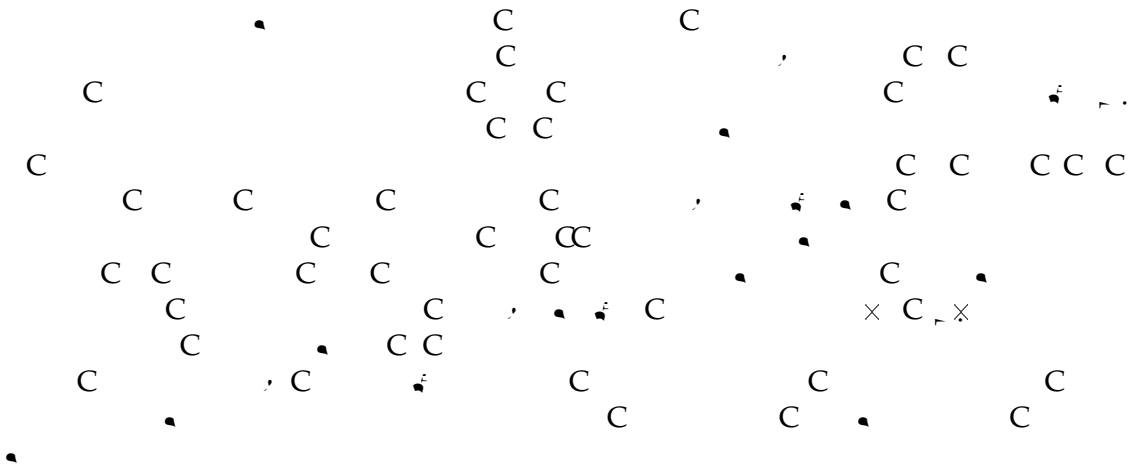
. . . . . .

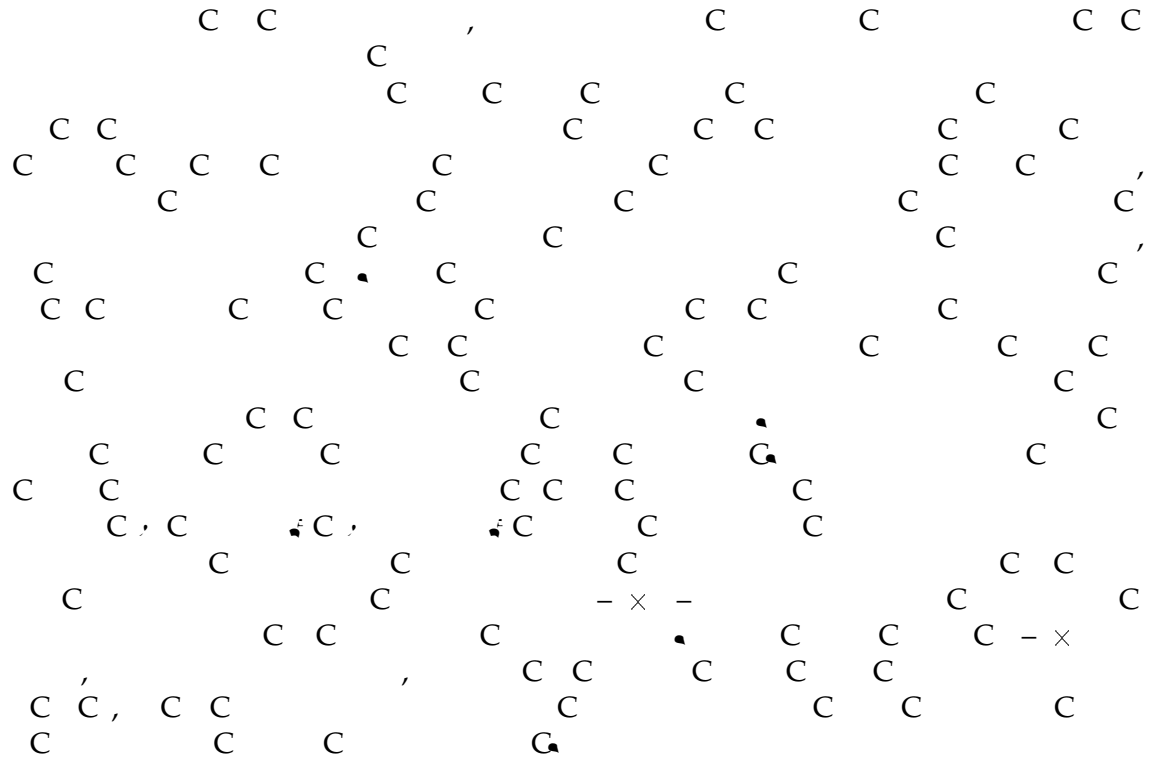
C

C

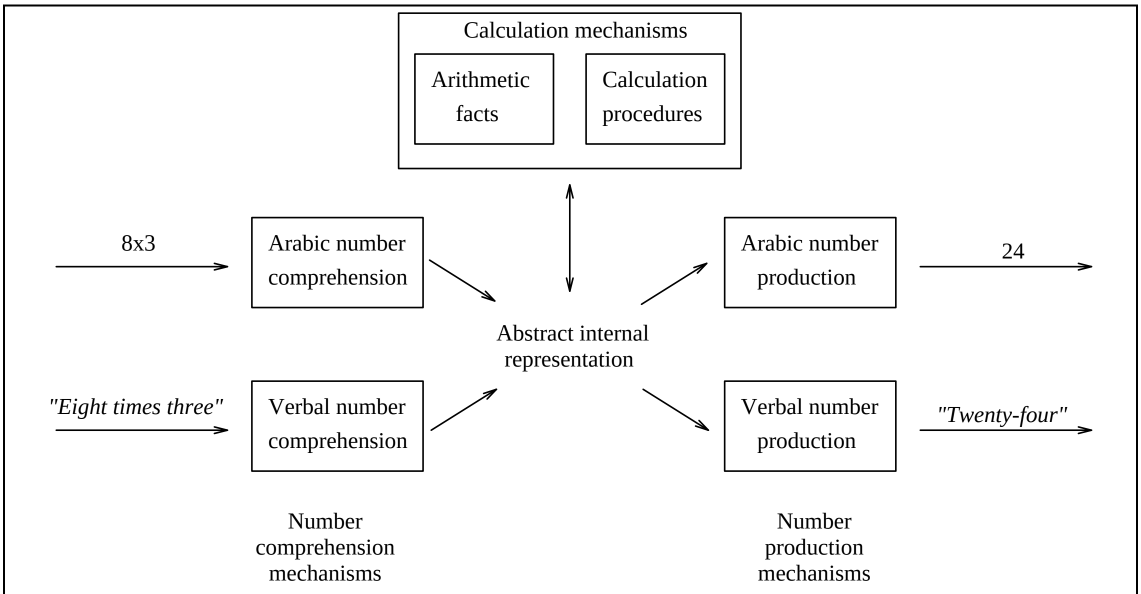
C C C .

Introduction





1.2 Part II—Multicolumn multiplication

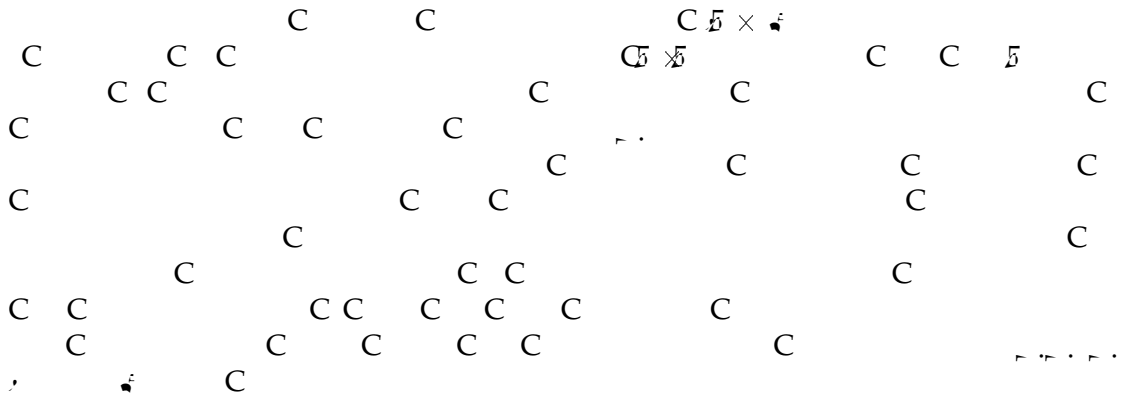


Calculation mechanisms
Arithmetic facts
Calculation procedures
8x3
Arabic number comprehension
Abstract internal representation
Arabic number production
24
"Eight times three"
Verbal number comprehension
Verbal number production
"Twenty-four"
Number comprehension mechanisms
Number production mechanisms

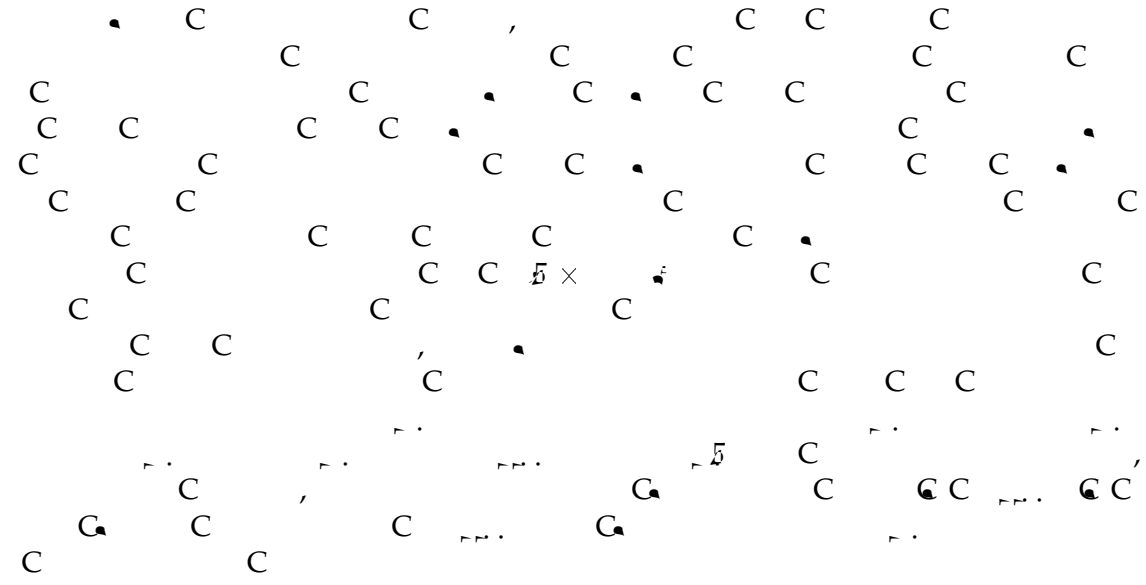
Figure

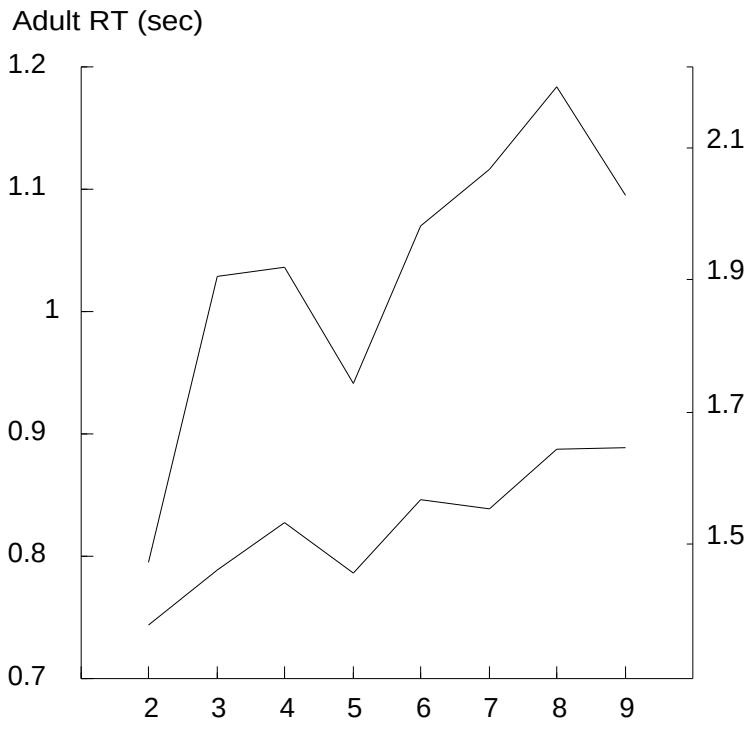
Menta, Arith etic

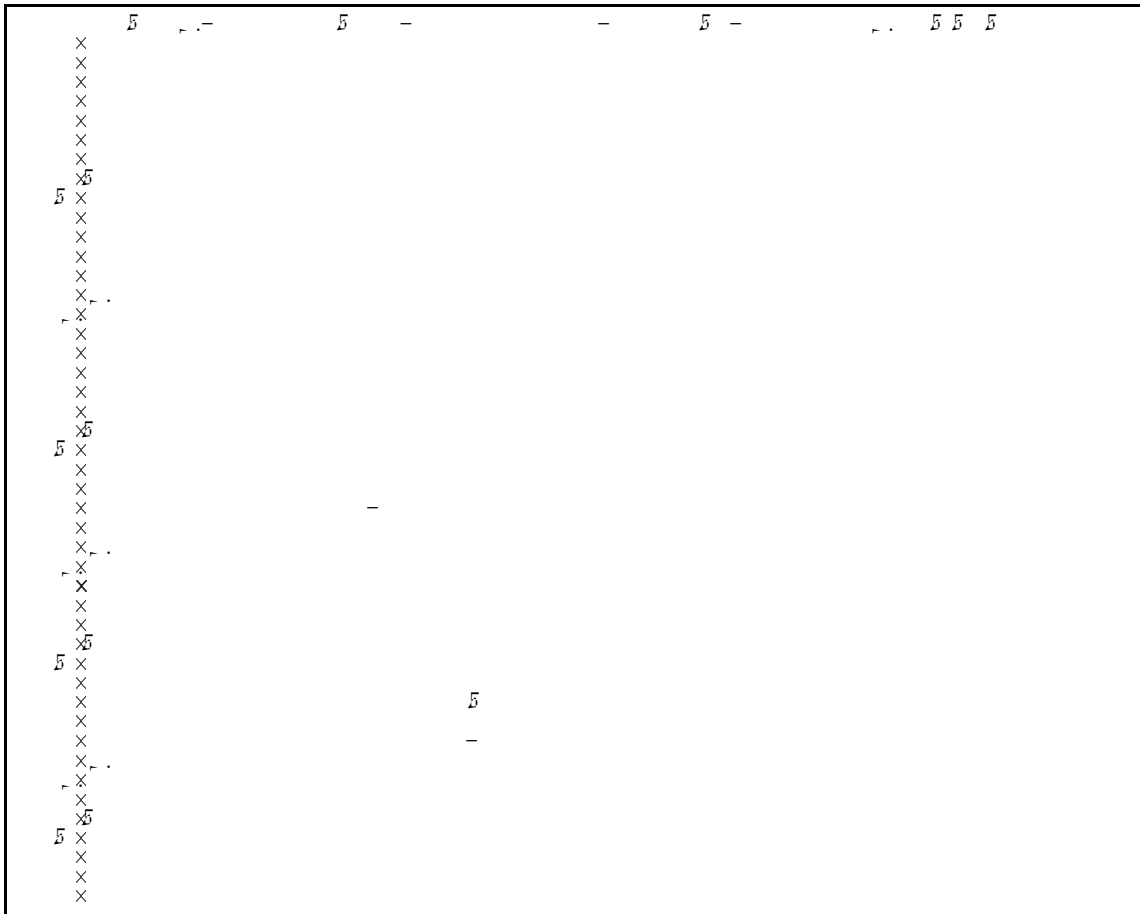
Me ory for Arith etic Facts



2.1 Phenomena

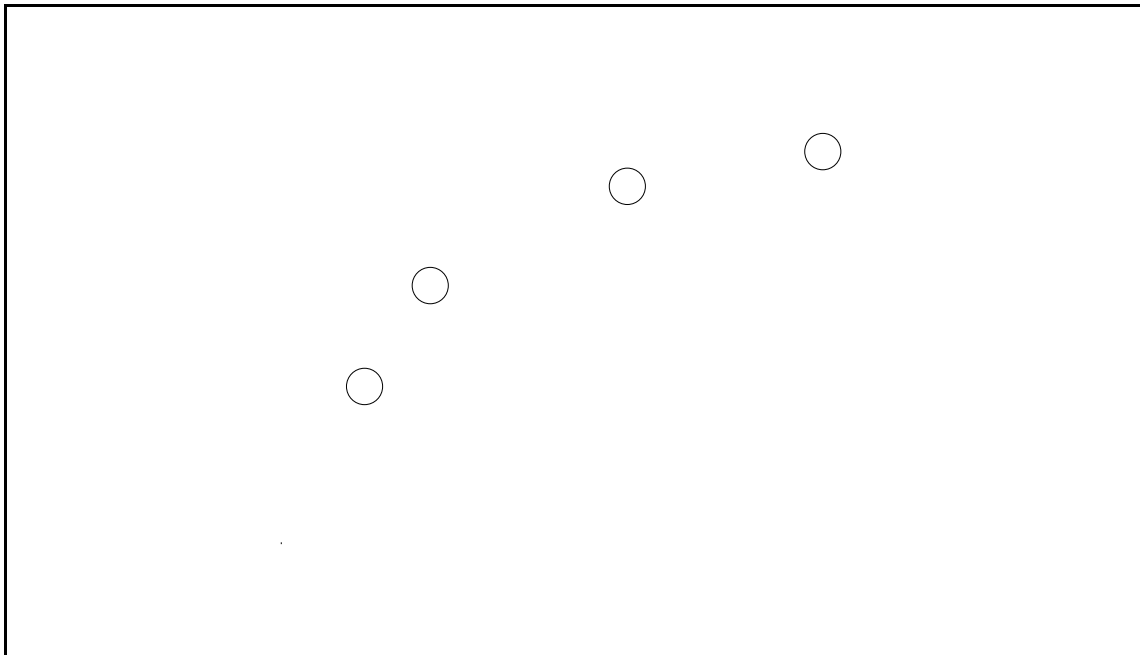



Adult RT (sec)
1.2
1.1
1
0.9
0.8
0.7
2.1
1.9
1.7
1.5
2
3
4
5
6
7
8
9



C C C C C
C C C C . .

- C C C C C C C C C
  C C C × − C
  C C C × −
- C C C C C C C C C C
  C C C C C C C C $a \times b$
  C C C C C $a \pm 2) \times b$ C $a \times (b \pm 2)$
  × C C C C
- C C C C C C 5 C
  5 C CC C C C



C

Ru e based processing

C C C – ⋇ C ⋇

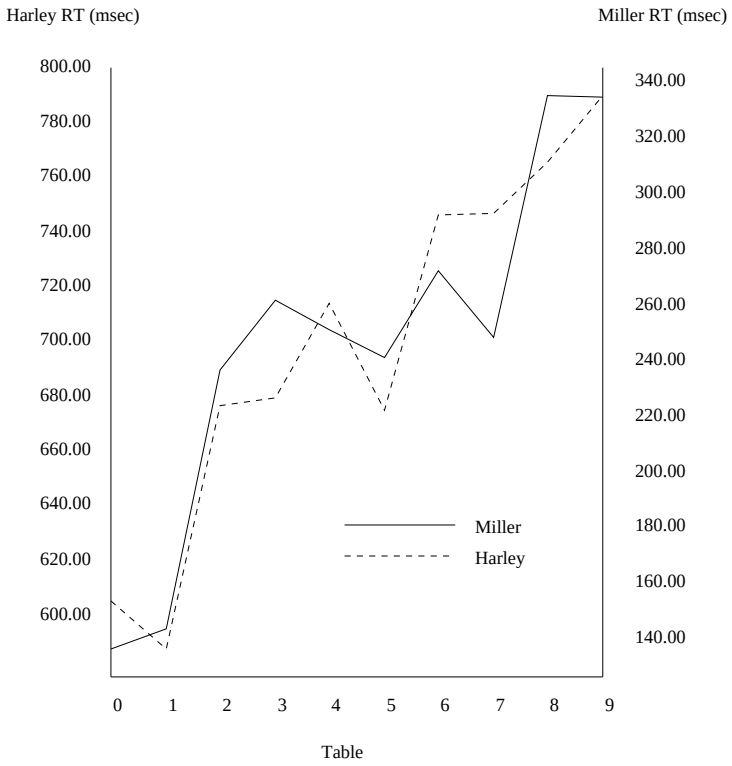
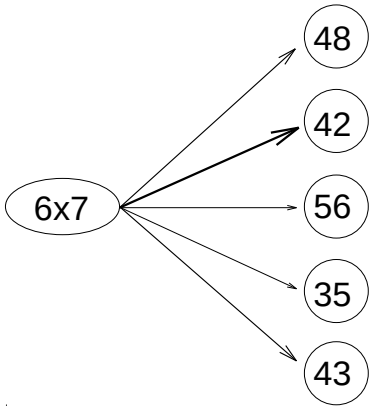


Figure C C C C C C



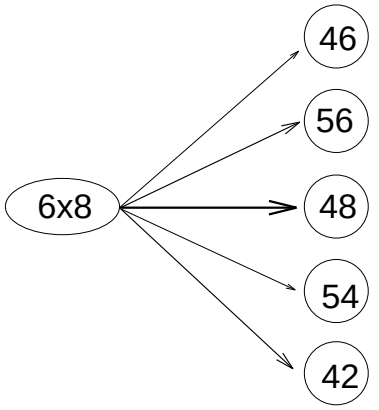


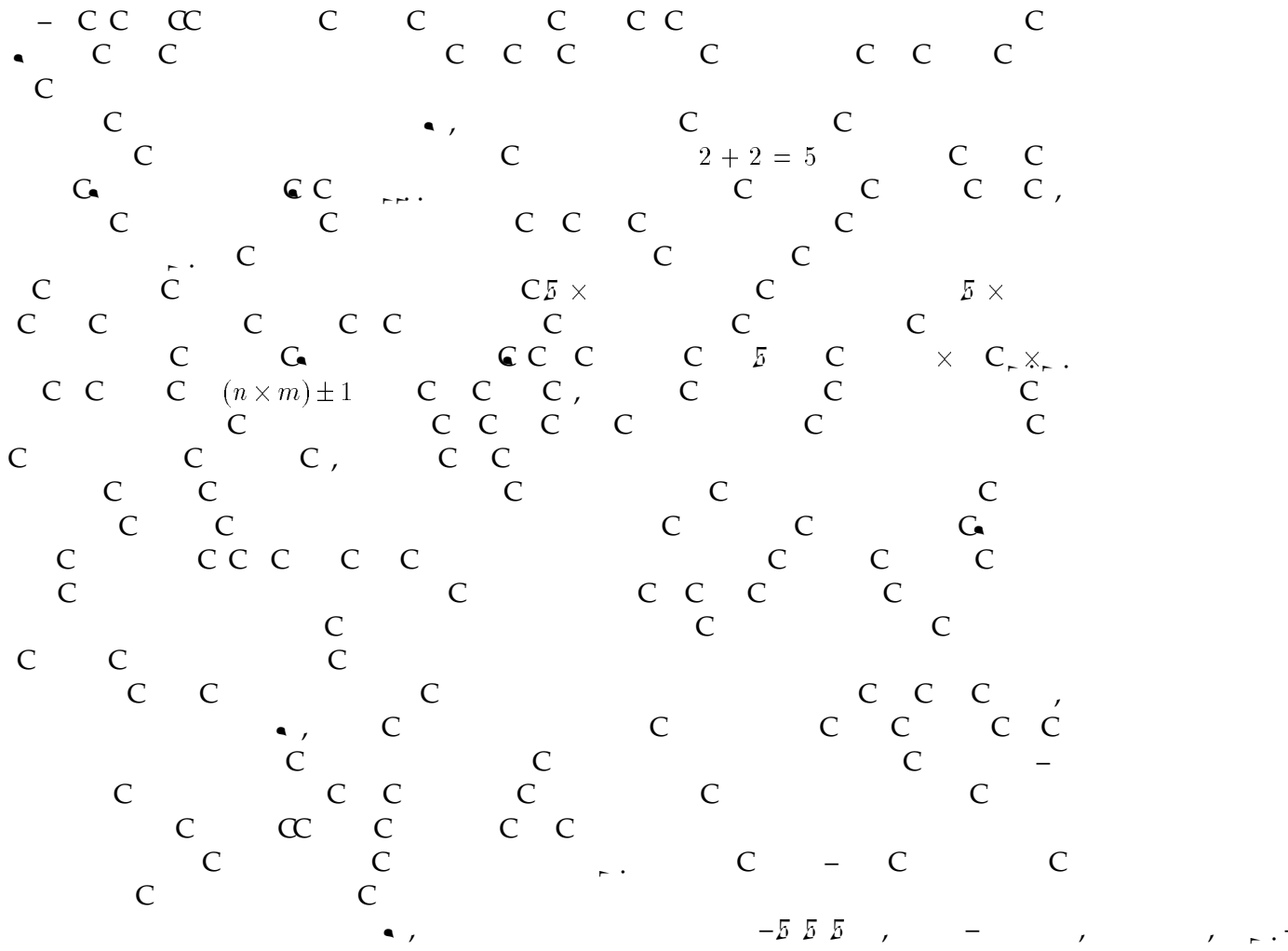
Figure C , C, C C C C
C C C C C C
C C C C C . .

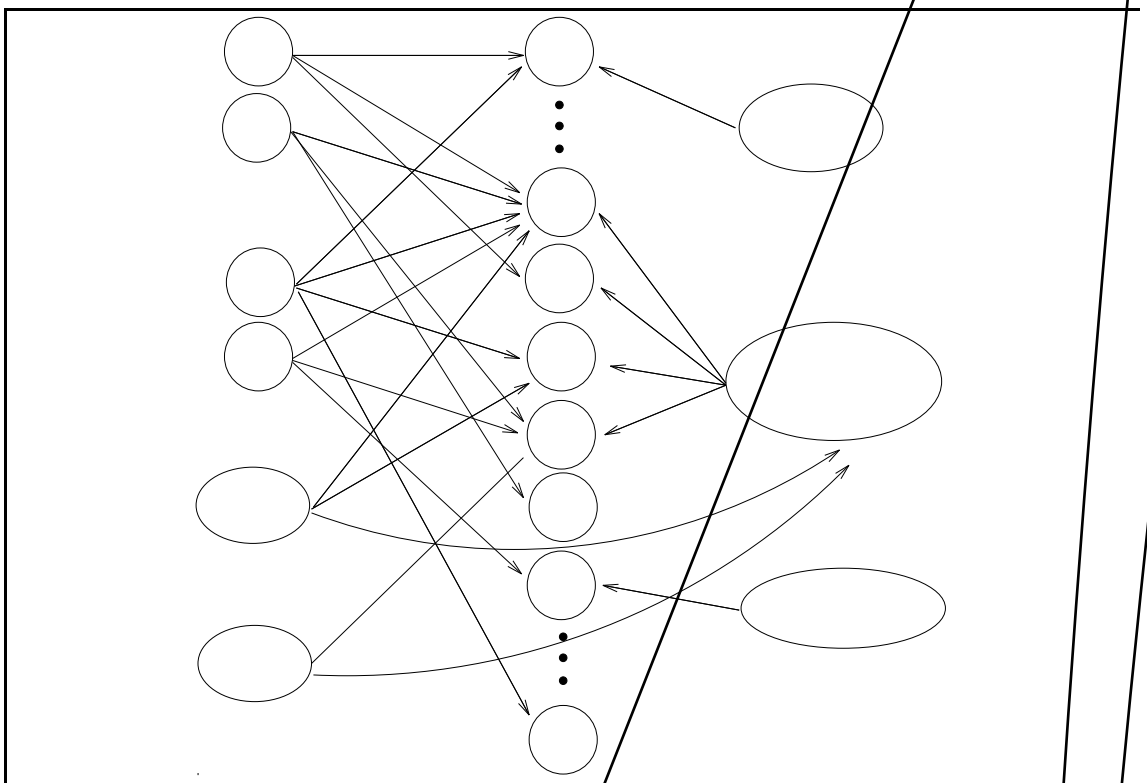
C C C C
,
C C C C C C C
C C C C C C ,
C C C C C C C C
C C , 5, 5 – , 5. C – 5

C C C C C
C C C ,
C C $n \ m$
C C $n \times m$

Reaction ti e

C C C C C C
C C C C C C



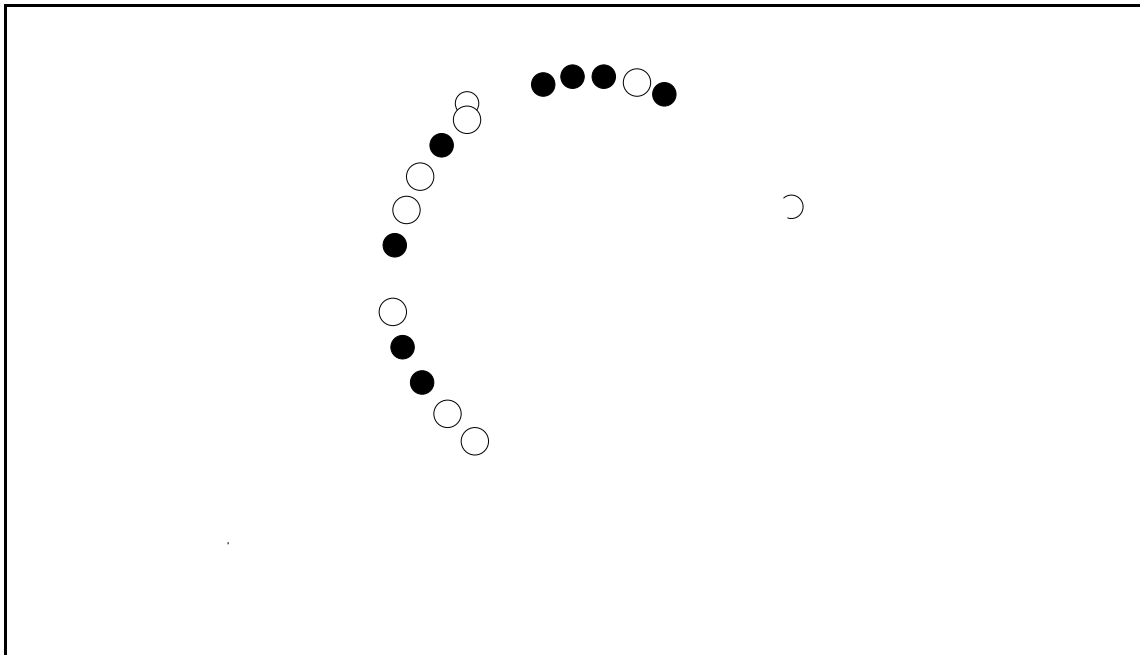


Errors

Retrieva,

Discussion

2.3 Previous connectionist models



Reca

C C

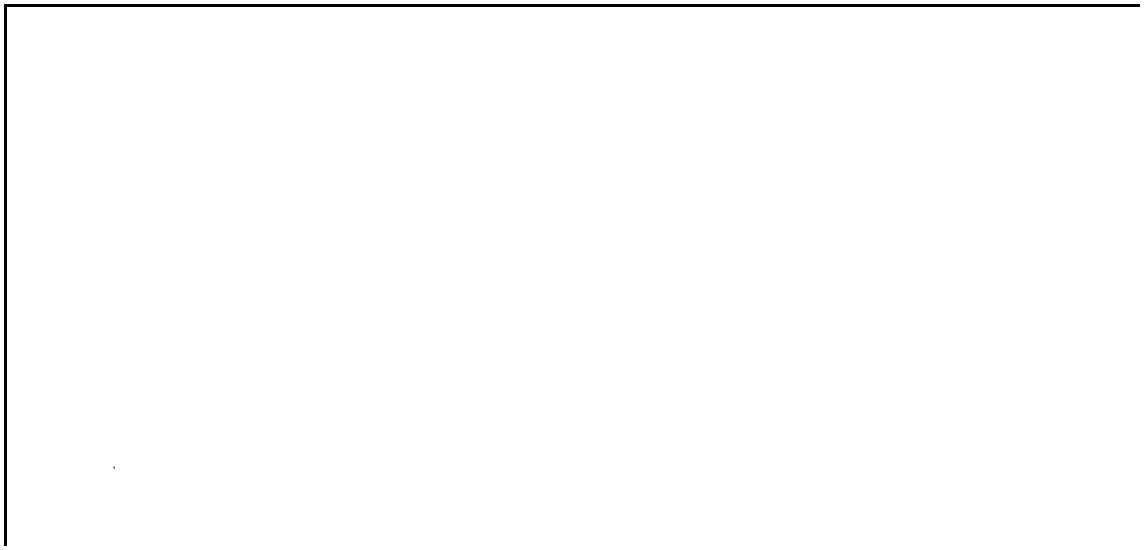
C C C

$$o_i(t+1) = \alpha \sum_j w_{ij} o_j(t) + \gamma o_i(t) + \delta f_i(0)$$

$o_i(t)$ C C i t C -1 $+1$

α C γ C $f_i(0)$ C

C δ



C C C C C–
C C C C
C C C C C–
C C C C C
C

Response ti es

C C , C , C, C
C C C C
C C C C C C
, C C – C , C
C C C C
C C – C
C C
C C
C C C C C
C C C C C
C C C C C , C C
C C C – – C C , C, C – – C
C , C, C C C

6 + C C 1.0 - –1.0 C

C

C C C 5

C 5

C C --

C , C 5

Tab e C C C C

C C C , C

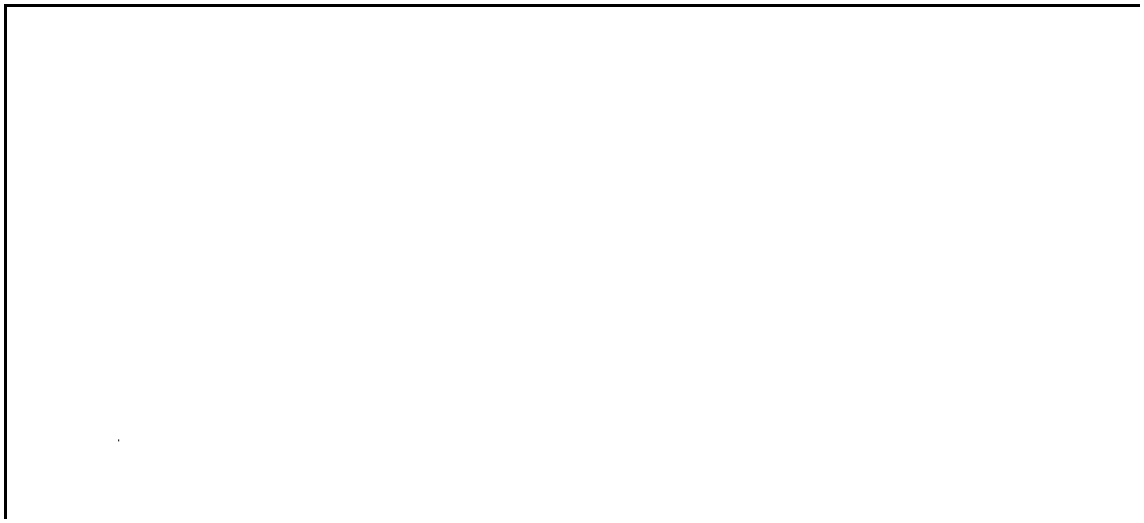
C C C C C
C C C C C C C ,
C C C C ,
C C C
C C , C C C C

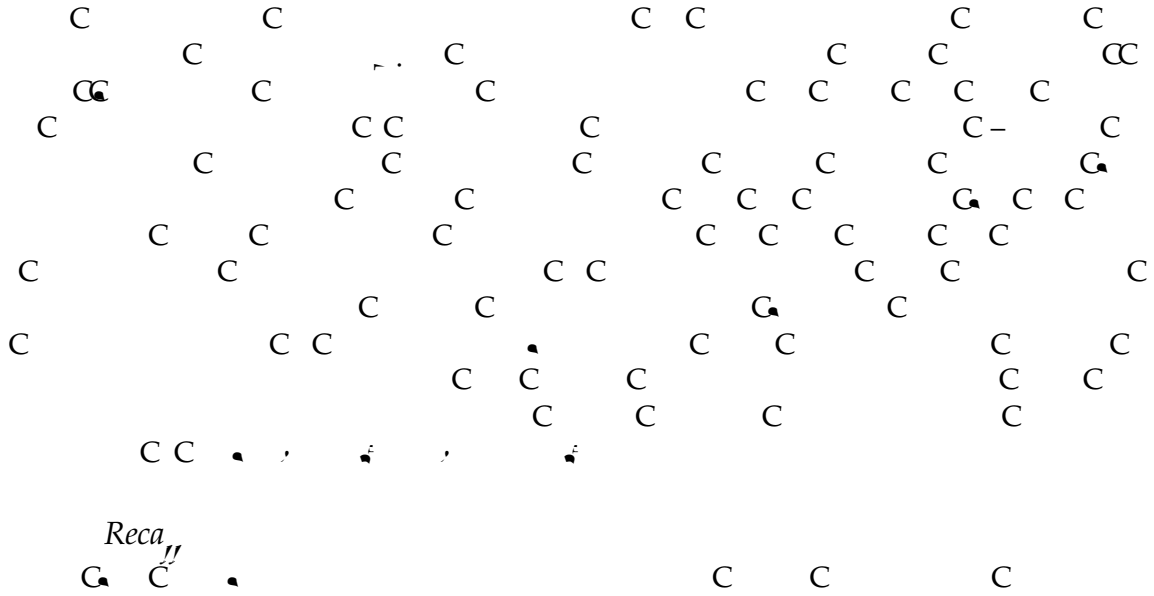
Da age

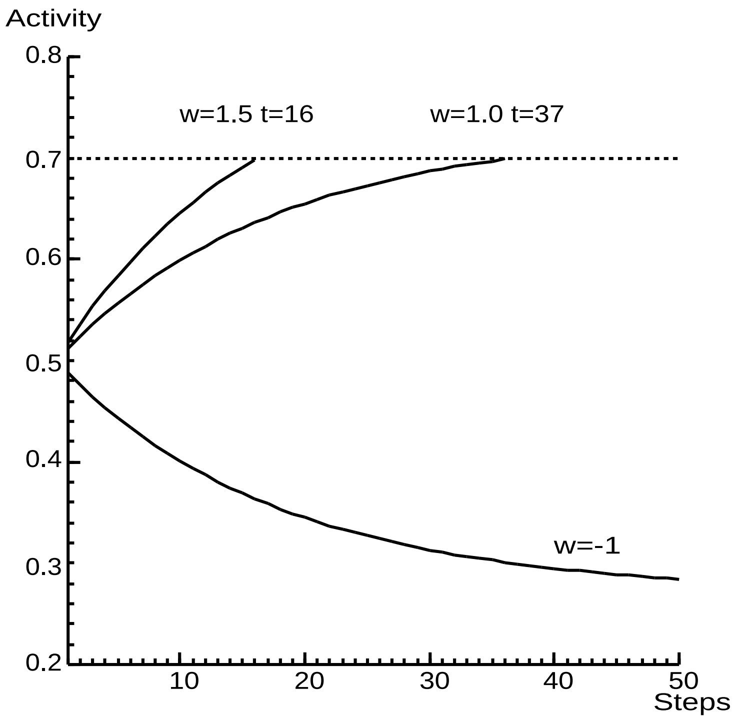
C , C
C C C
C C C C
C C C C C
C C C C
C C – C
C C – 5
C C C C .
C , C C C

C C C C CC C
C C C C C
C C C C C C C C C C

- C , C C C C C
C C C C C C
C C C C C C C
C C
- C C
C C C C C
C C C CC
- C C
C C C C C C
C C C
C C C
, , C , , C C C



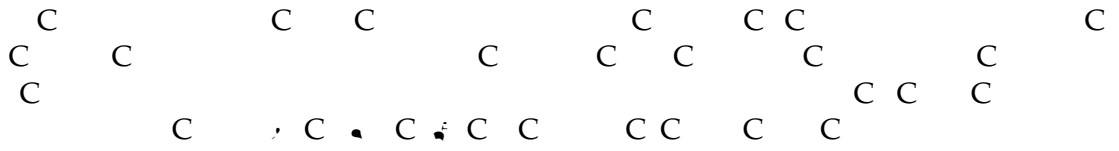




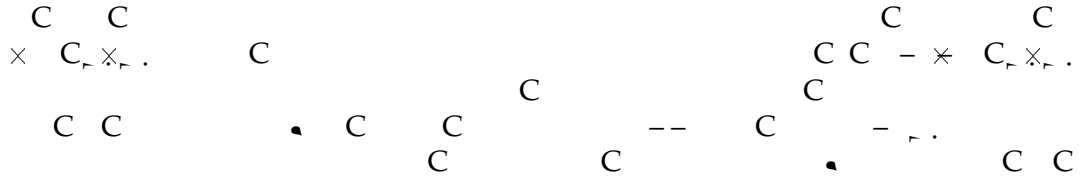
Figure

$$o_i = \frac{a_i}{\sum_j a_j}$$

i j

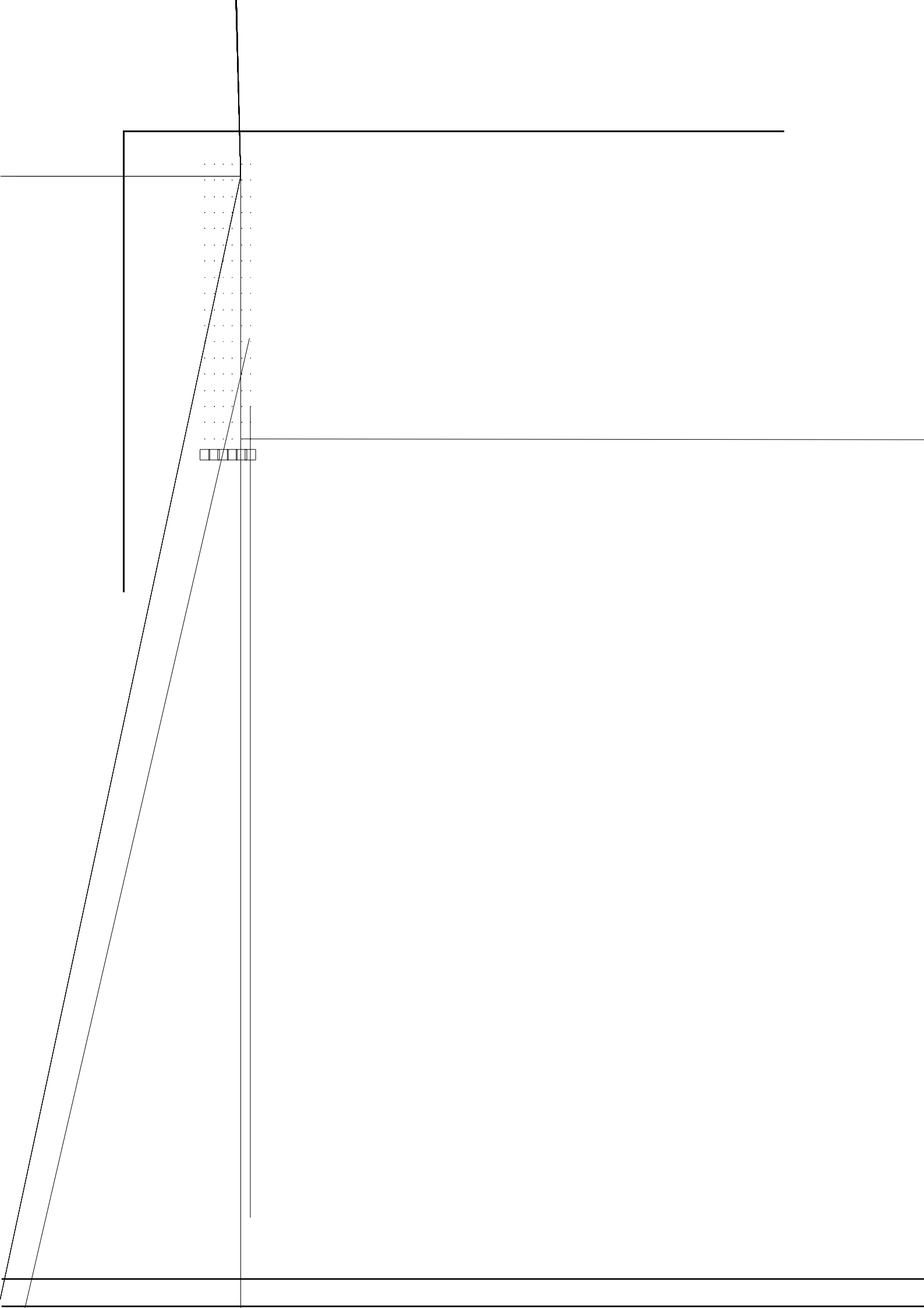


Training



C C C
C C ,
C C C C
C C C C C
C C C C C

3.2 Simulations for 2×2 to 9×9



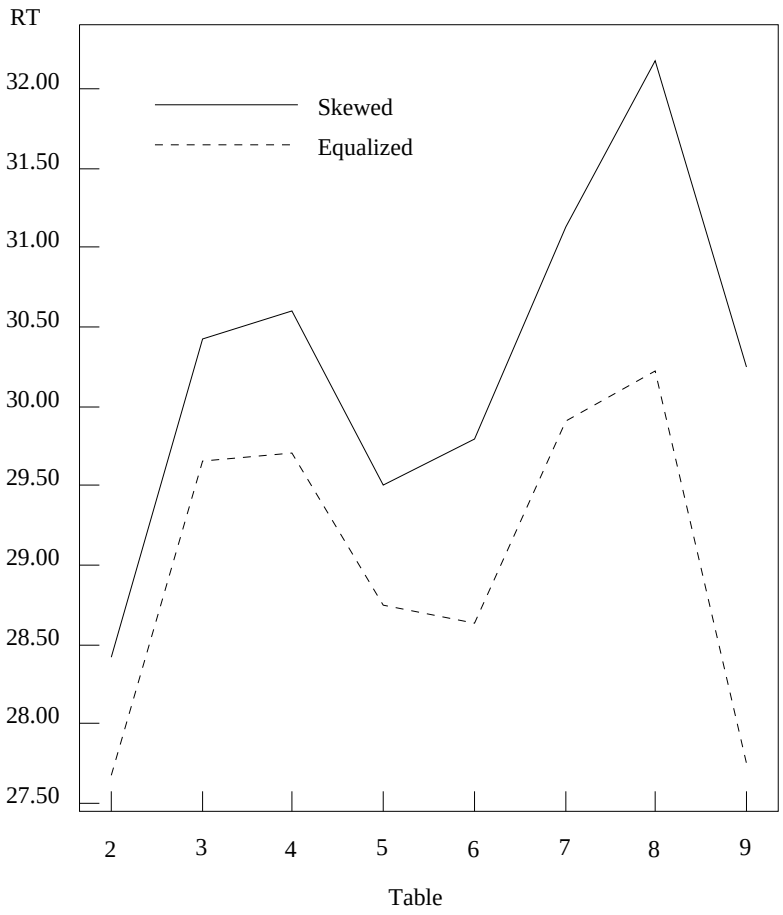
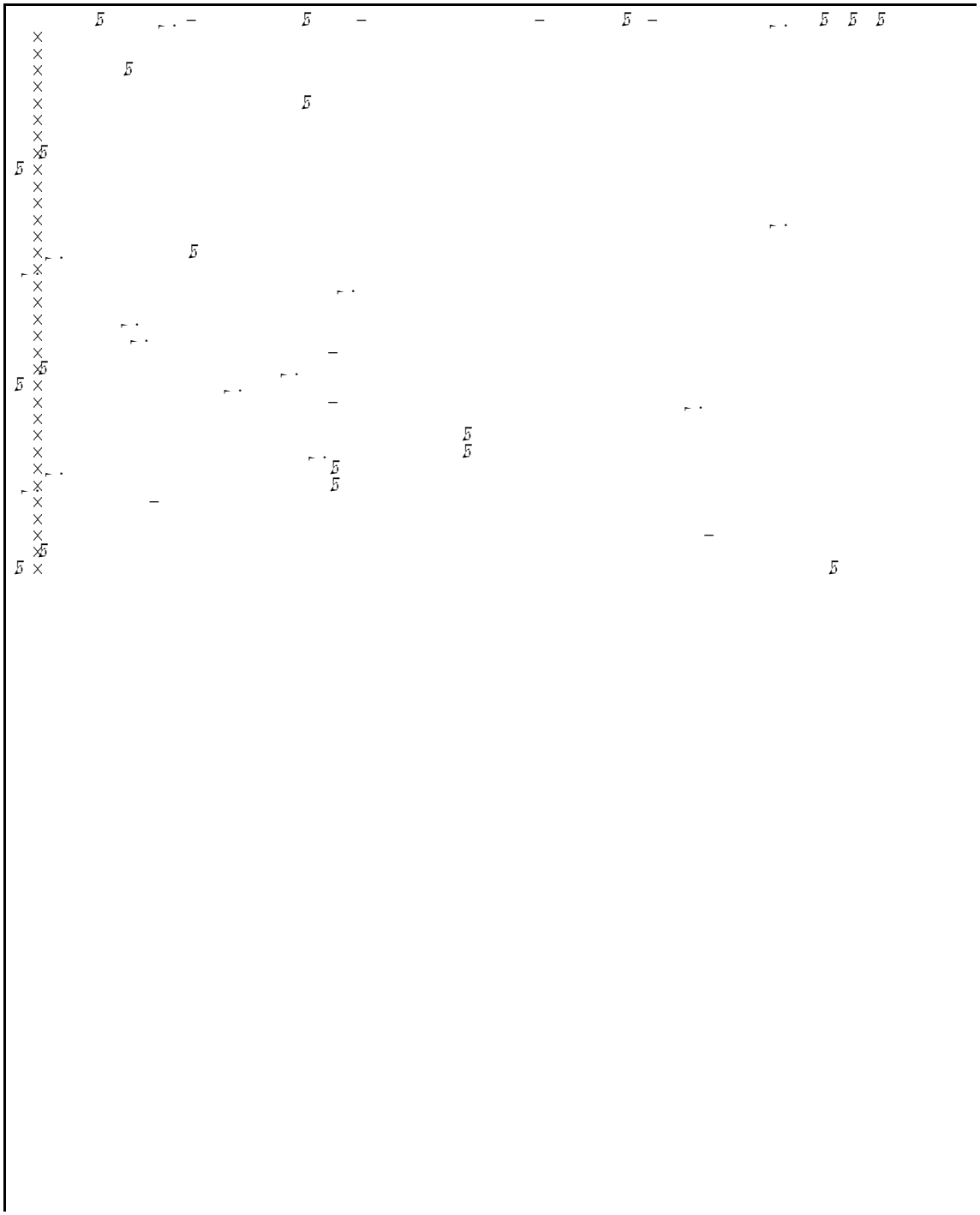
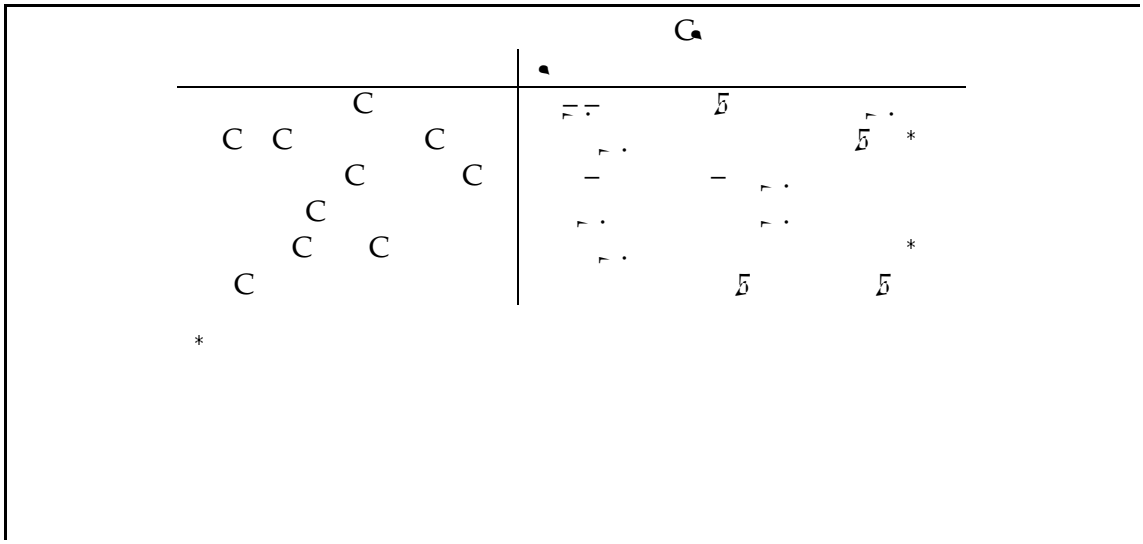
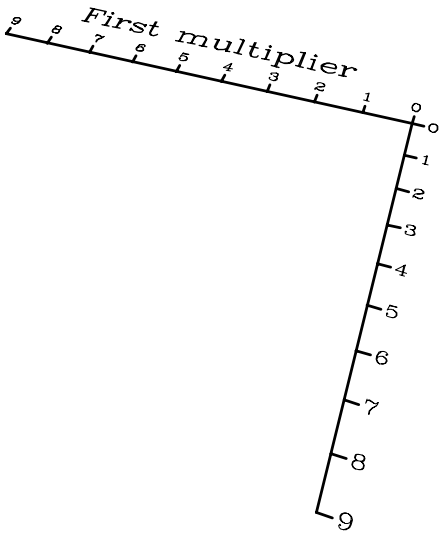


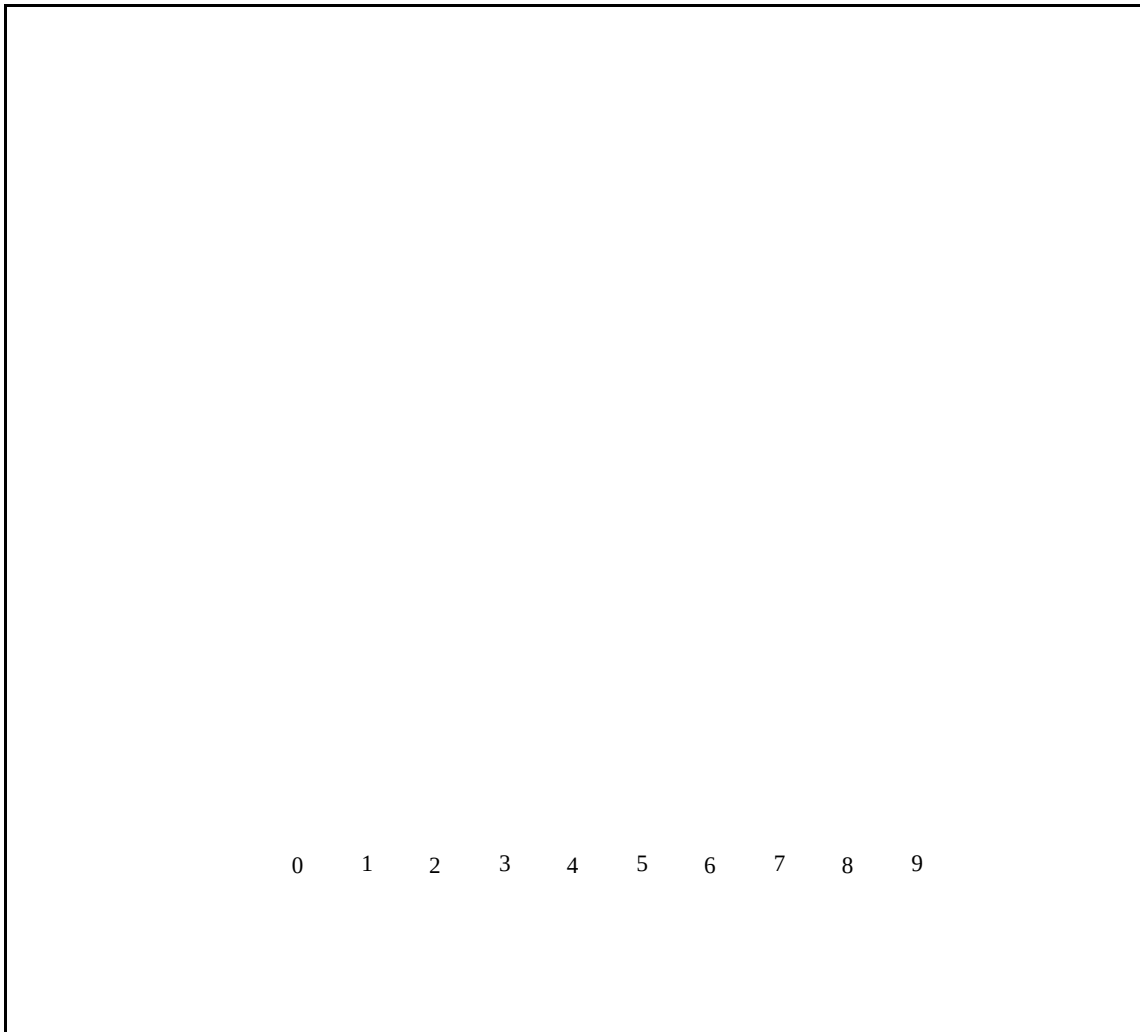
Figure C C C C C
C C C – G –

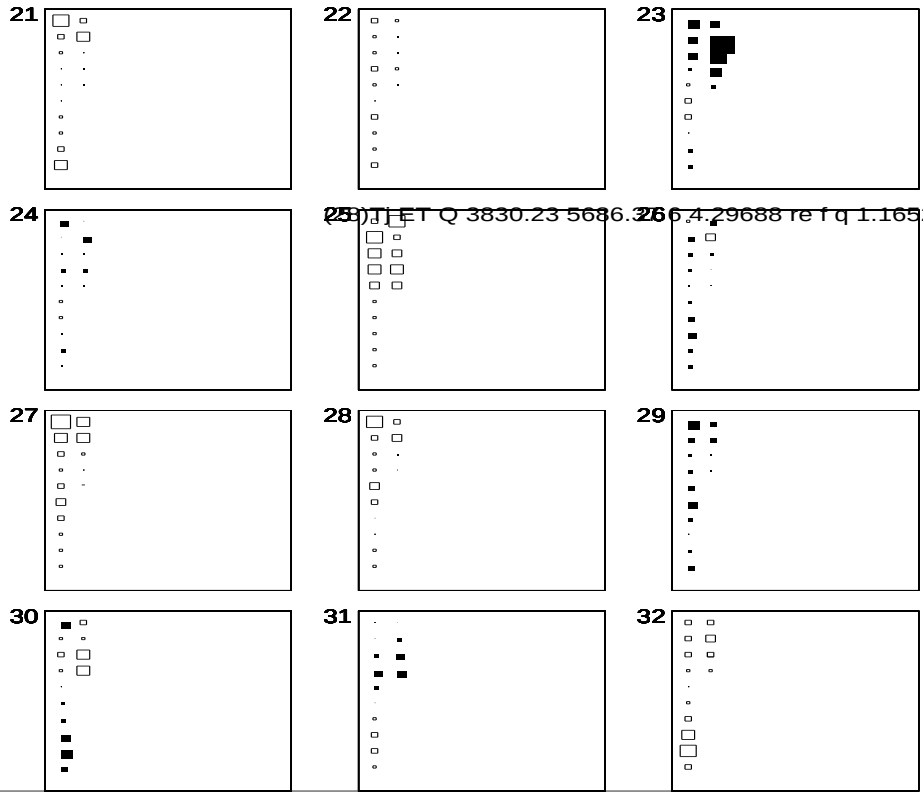
C C C C
C C G C5 C C C C
C C C G 5 ×5
C C
C C C C – G C
C C C C



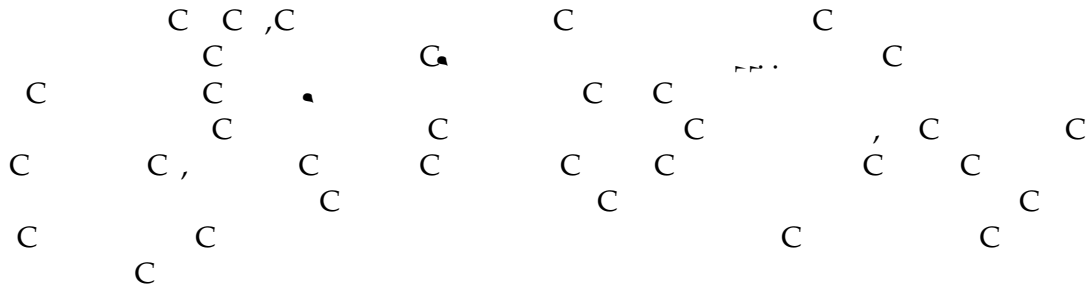



First multiplier
9 8 7 6 5 4 3 2 1 0
0 1 2 3 4 5 6 7 8 9


0 1 2 3 4 5 6 7 8 9



(28)Tj ET Q 3830.23 5686.326.4.29688 re f q 1.16527 0 0 0.834726 0 0 cm 3163.4



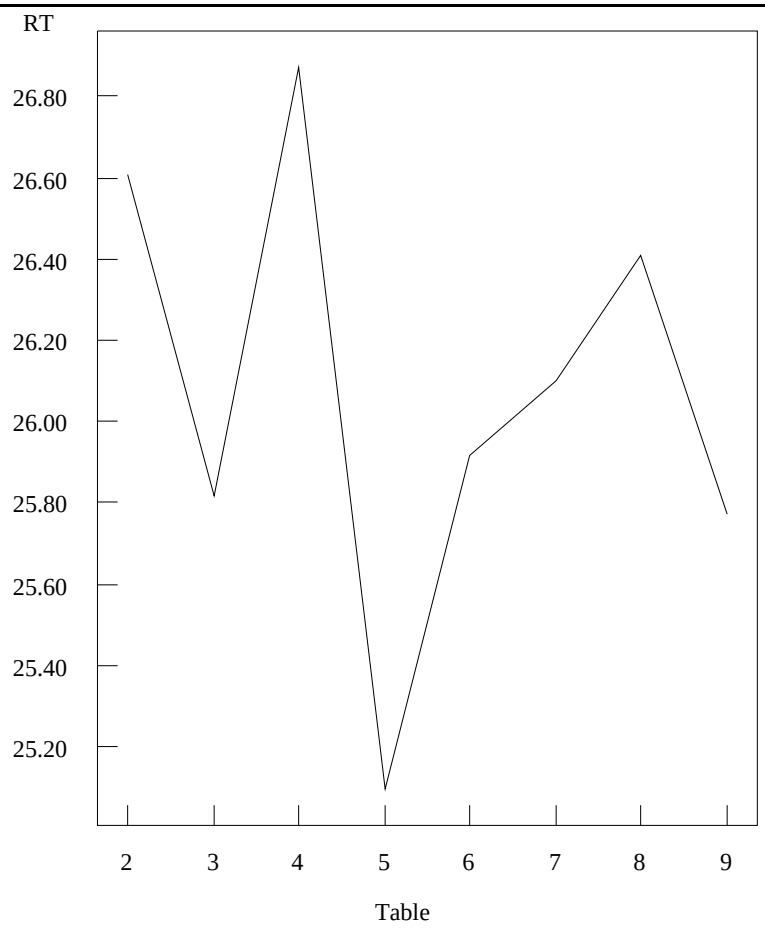


Figure C C C – G C

C , C C C – C
C C C
C C C C
C C – C C . C –
C C– C – – C C C
× C C

$$\text{frequency} = \frac{180 - \text{product}}{360}$$

C C
C C C C C – .

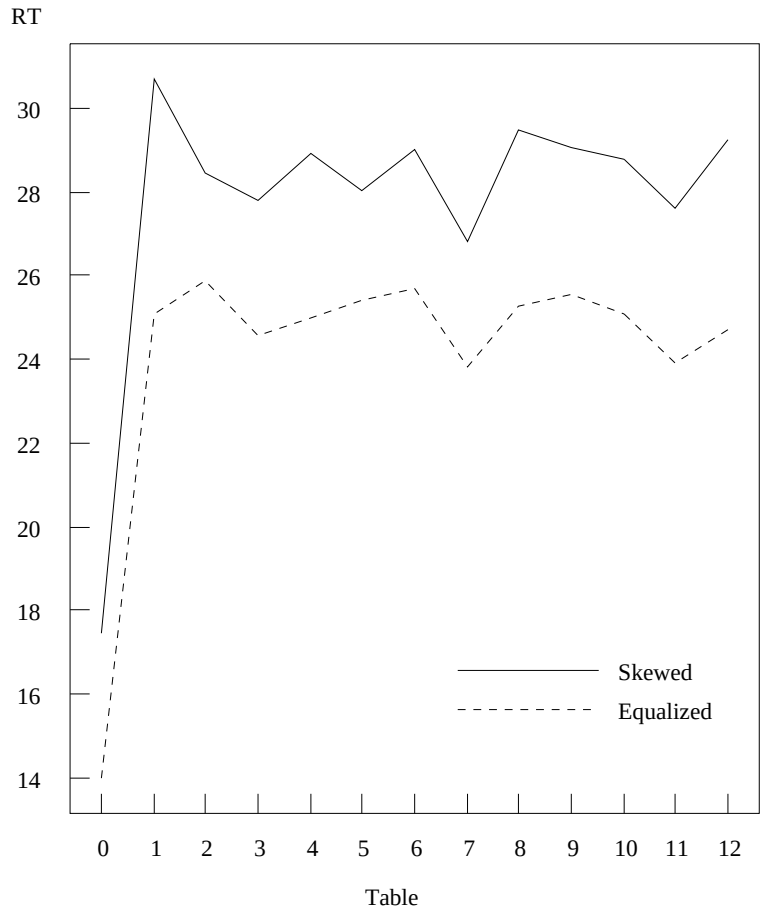
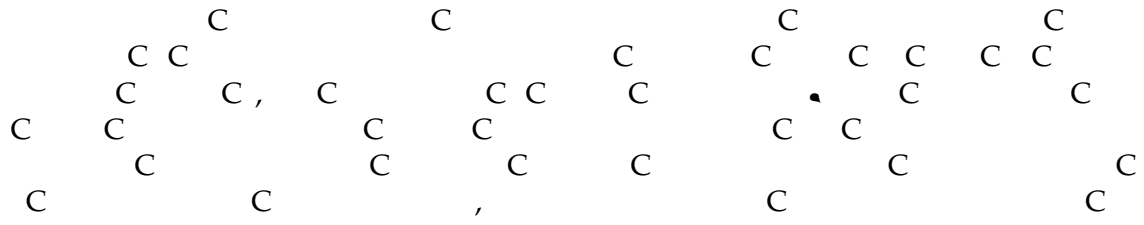


Figure C C –

Tab e C C C C – × C . C C

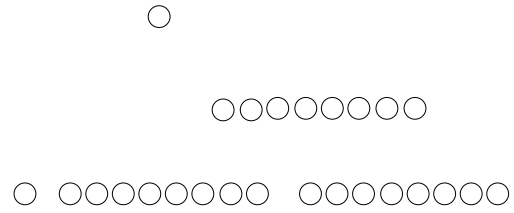
C C , C

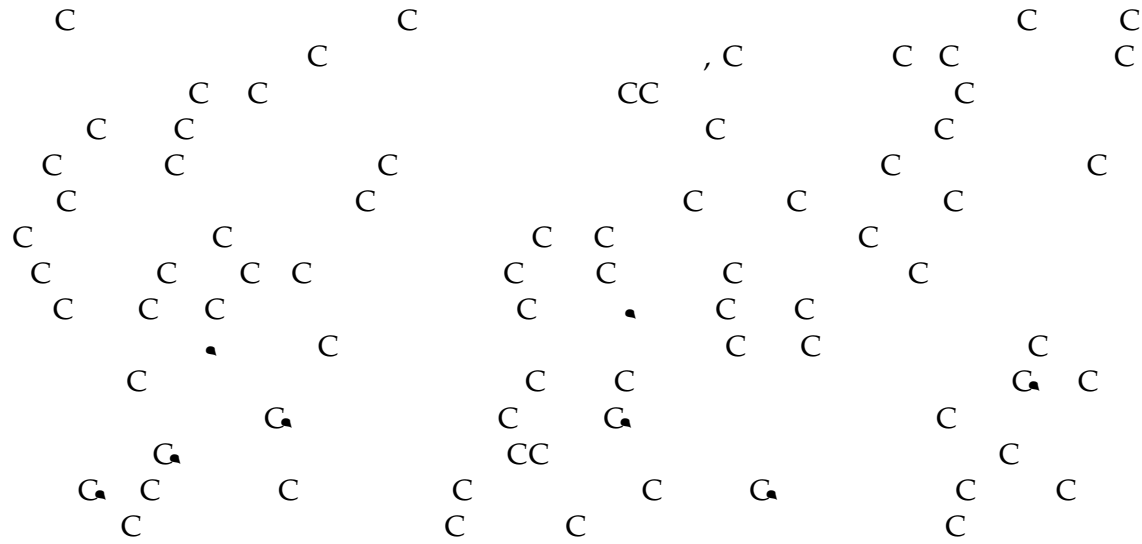
C , .



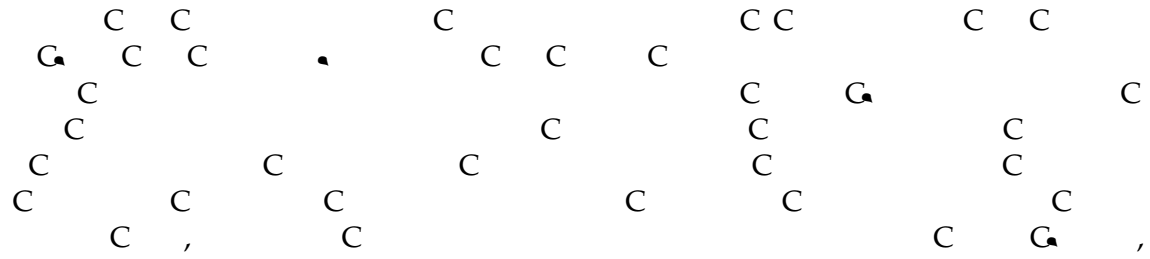
Choice of output representation

True False



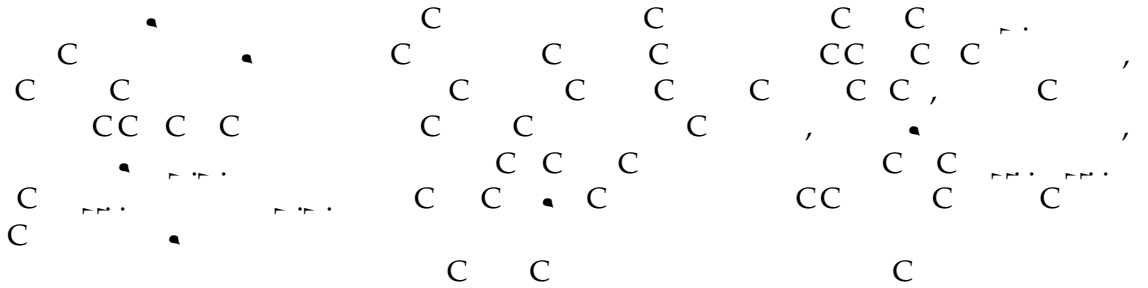



81
72
64
63
56
54
49
48
45
42
40
36
35
32
30
28
27
25
24
21
20
18
16
15
14
12
10
9
8
7
6
5
4
3
2
1
0
?
3x3



Mu¸tico¸u n Arith etic

Sy bo ic Accounts of Arith etic

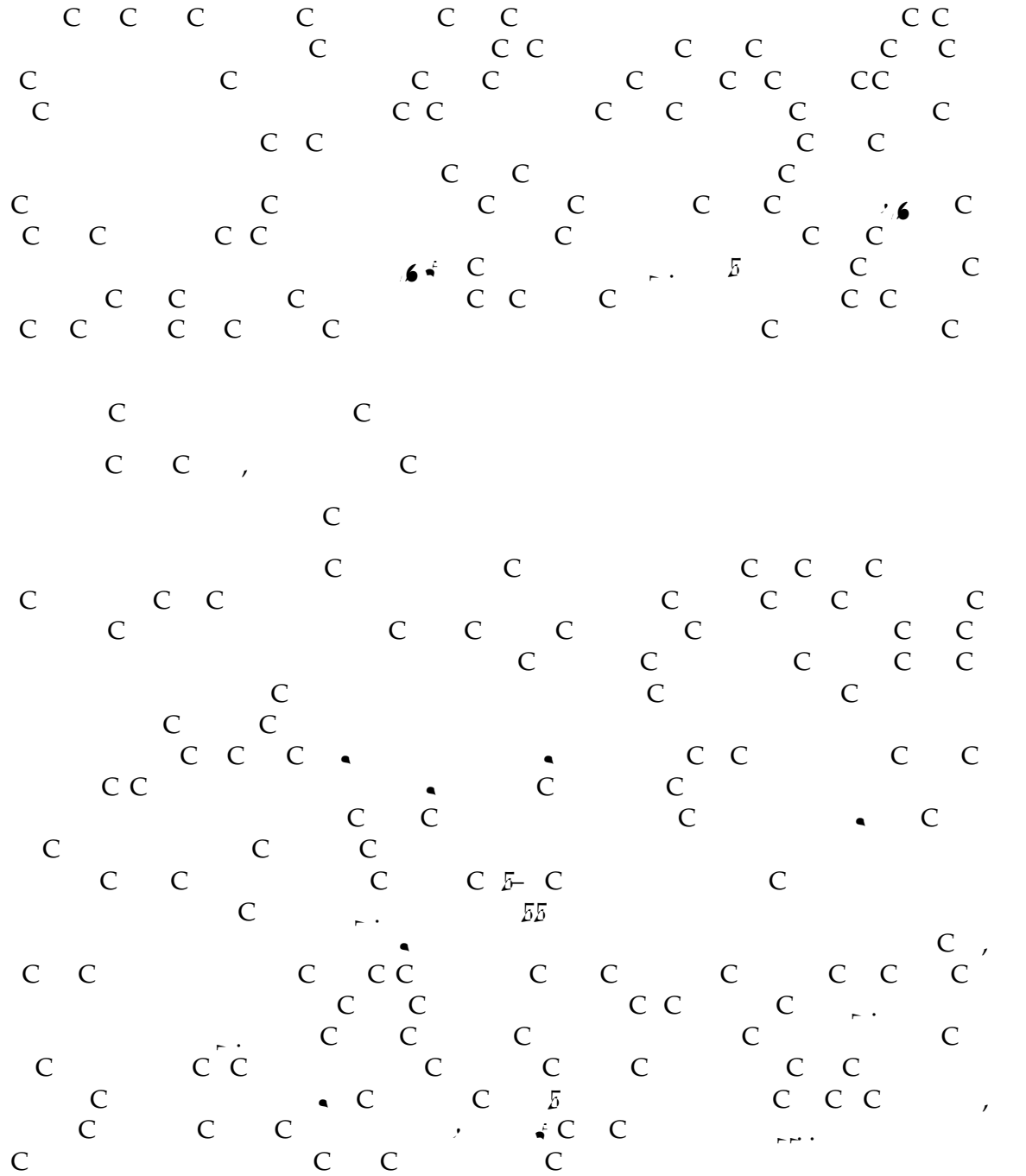


	Conditions		*Actions*
INTO:	[processmult]	⇒	readintandb();
SM:	[t ?t] [b ?b] [c ?c]	⇒	do_calc();
NX:	[next_top]	⇒	[processmult] shift_top_left();
WM:	[result ?u] [carry ?c]	⇒	writedown(); [next_top]
CC:	[no_more_top]	⇒	checkcarry(); [checkbottom] [addzero]
CB:	[checkbottom]	⇒	check_bottom();
FI:	[none_left]	⇒	[stop]
NB:	[no_more]	⇒	endmult(); [startadd]
CO:	[startadd]	⇒	readincolumn();
DA:	[column ?len ?dig]	⇒	do_add();
ML:	[next_left]	⇒	[startadd] moveleft();
WA:	[u ?u] [c ?c]	⇒	writeadd(); [next_left]
CA:	[no_more_digits]	⇒	checkadd();
AZ:	[addzero]	⇒	add_zero();

Tab e

4.2 Models

[processmult] C C C C C
C C C
C C C C



Learning by induction

```
(Problem 1)                        Object    is a subtraction probʻe
(Column 2) (Column 3)              Objects        are coʻu   ns
(Part 1 2) (Part 1 3)              The coʻu   ns are part of the probʻe
(First 1 3)                        Object    is the ʻeft   ost object
(Adjacent 1 2 3)                   The coʻu   ns are adjacent
(Cell 4) (Cell 5) (Cell 6)         Objects       are ceʻʻs
(Digit 4) (Digit 5)                Objects   and   are digits
(Blank 6)                          Object    is a bʻan   ceʻʻ

Sub1Col(C) OR
1. [And (Digit T) (Part-of T C) (First T C)
        (Digit B) (Part-of B C) (Middle B C)
        (Ordered C T B) (Adjacent C T B)
        (Value-of TV T) (Value-of BV B) (LessThan TV BV)
   ->   (Borrow C)

2. [And (Digit T) (Part-of T C) (First T C)
        (Digit B) (Part-of B C) (Middle B C)
        (Ordered C T B) (Adjacent C TB)
        (Value-of TV T) (Value-of BV B)
        (Less-Than-or-Equal BV TV)
   ->   (Diff C)

Diff(c) AND
1. [And (Digit T) (Part-of T C) (First T C)
        (Digit B) (Part-of B C) (Middle B C)
        (Cell A) (Part-of A C) (Last A C)
        (Ordered C T B) (Adjacent C T B) (Ordered C BA)
        (Value-of TV T) (Value-of BV B)
        (AbsoluteDifference TV BV AV)
   ->   (Write AV A)
```

Tabʻe C C C C . C , C C C C C .

The i passe repair process

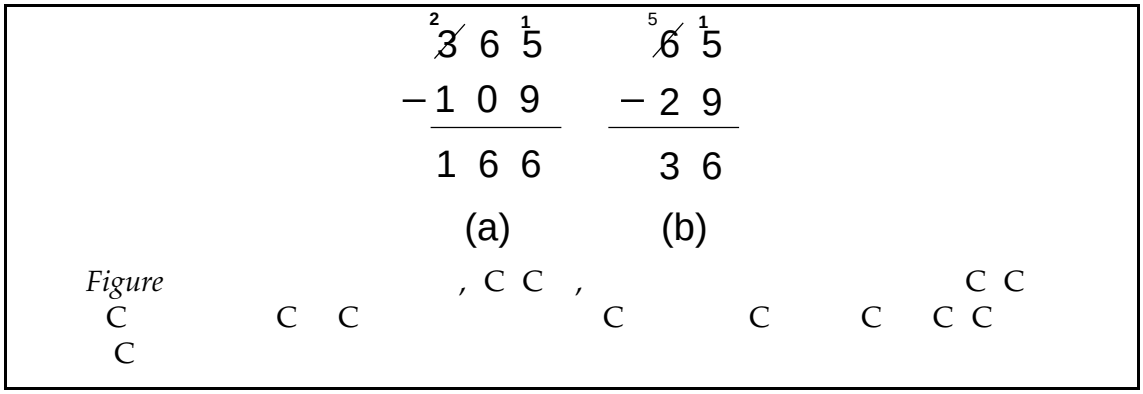
C C C C C C C C C C C C

C C C C C C C C C C C C C C C

C C C C C *i passe* C C

C C *repairs*

C C C C C C C C C C C C C C C C



Figure

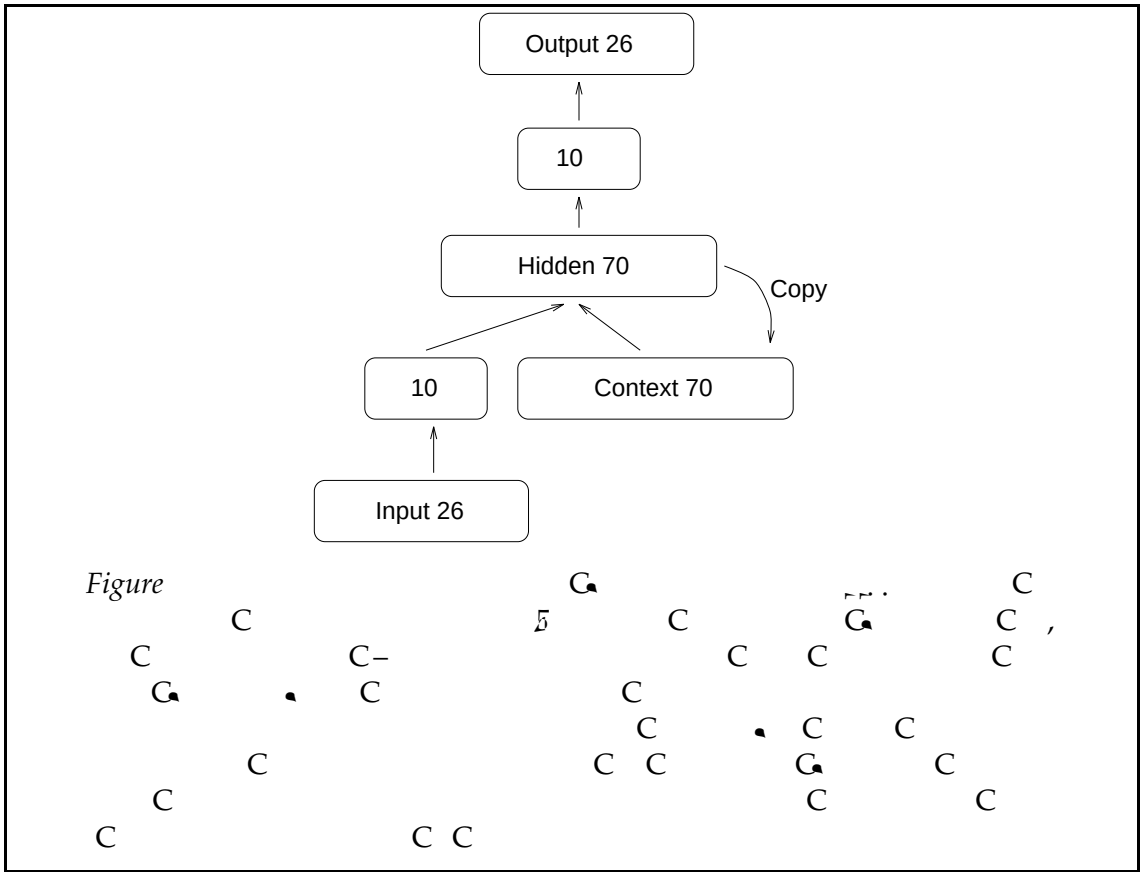
E pirica, adequacy

C C C C
C C C C C *patch* C
C C C
C C C
C C
C C C C C
C C C C
C C

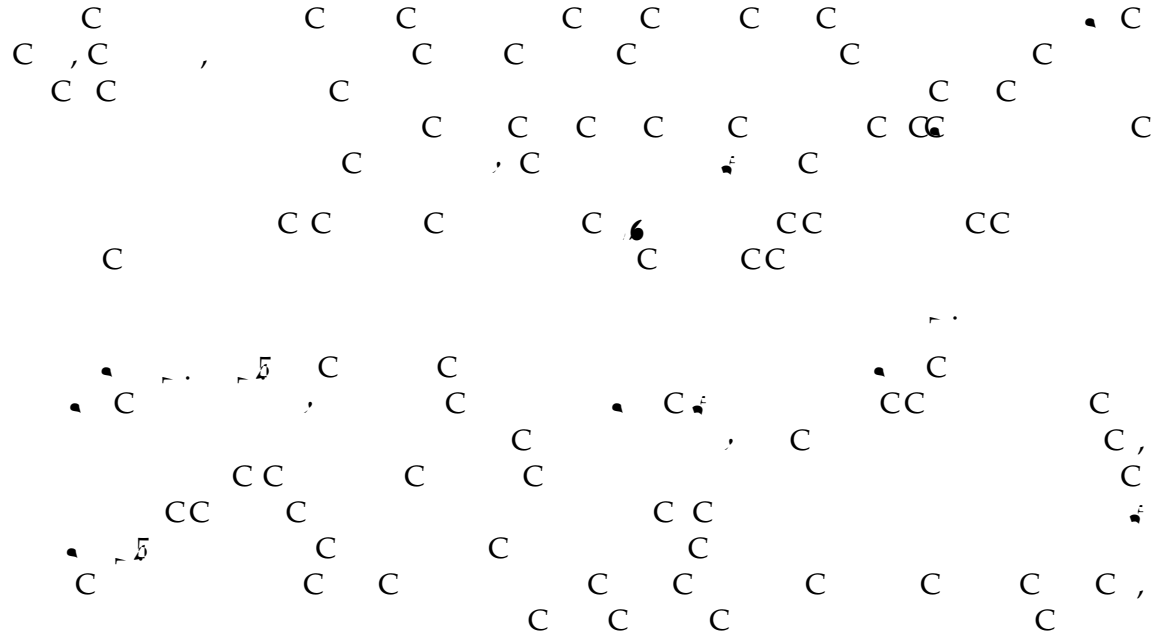
C C C C
C C C C
C C ,
C C , C C C C
C C C C C ,
C C C C C
C C C C
C C C C C ,
C C C C CC
C C C
C C ,
C C C C C C

C C C

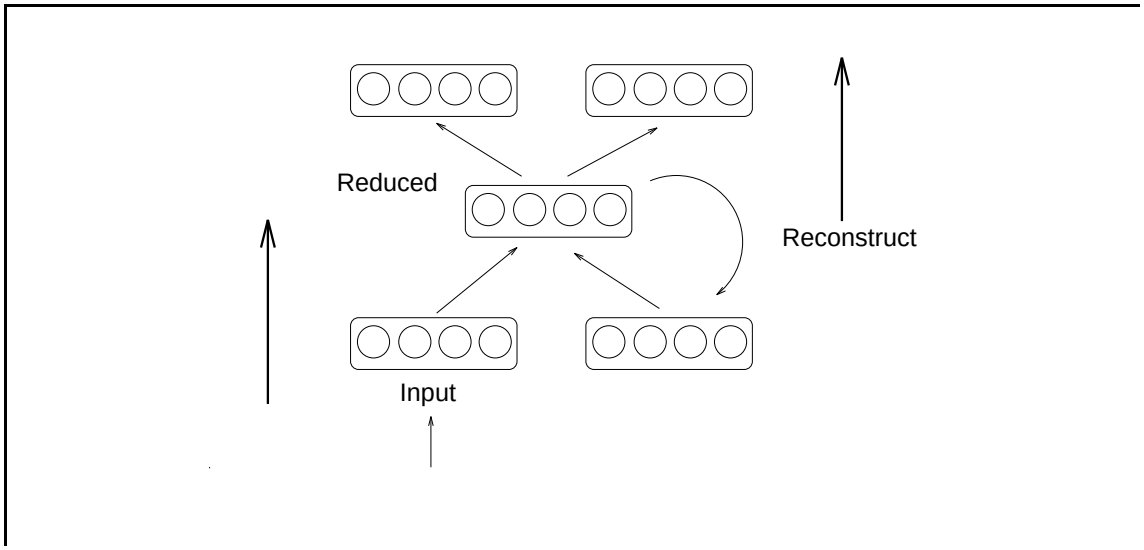
Hidden unit recruit ent



Figure

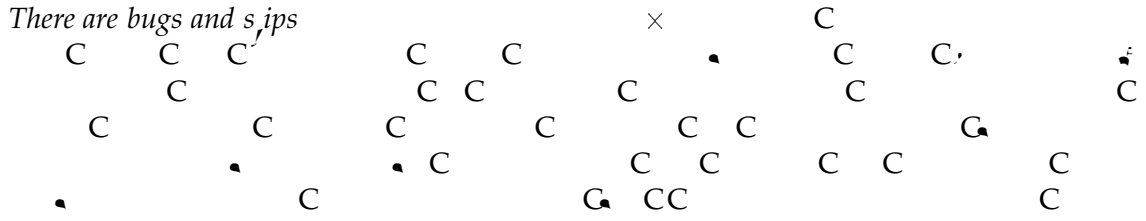


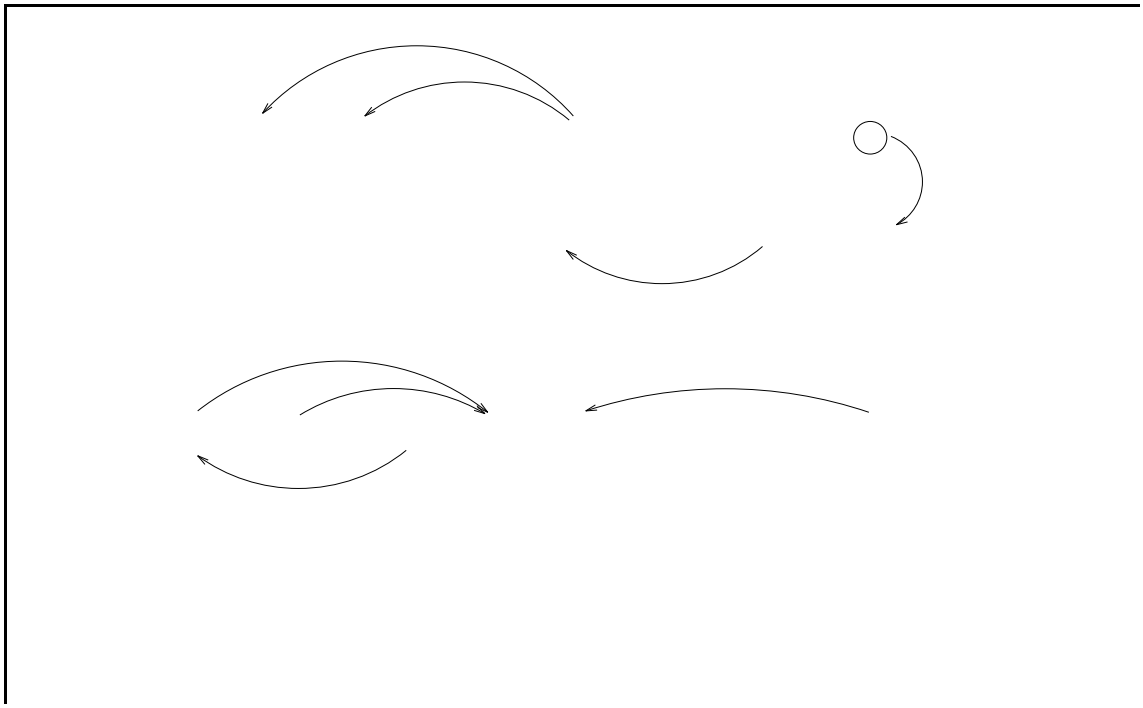
C C , C C
C ,
C C C C

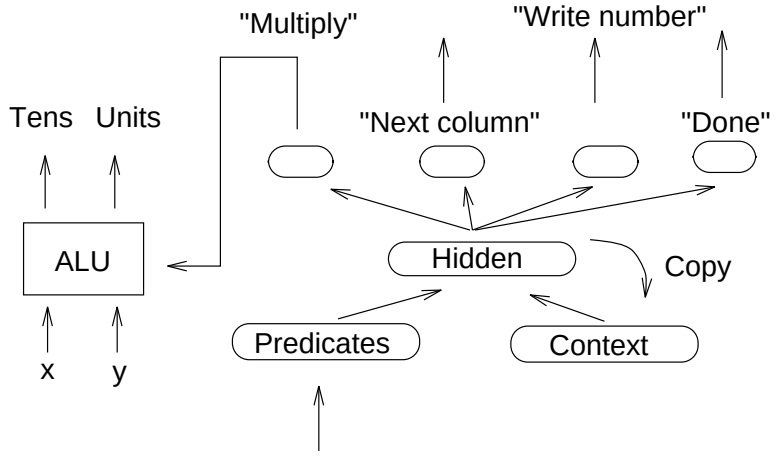

Reduced
Reconstruct
Input

C C C C
C C C C
C
, C C C C C
C C C C C C C C C C C C C C
C C C C C C C C C
, C C , C C C C
C C C C C
C C C C C C
C C C
C C C C C C C C C
C C ,
C
C C C , , ,
C C C C

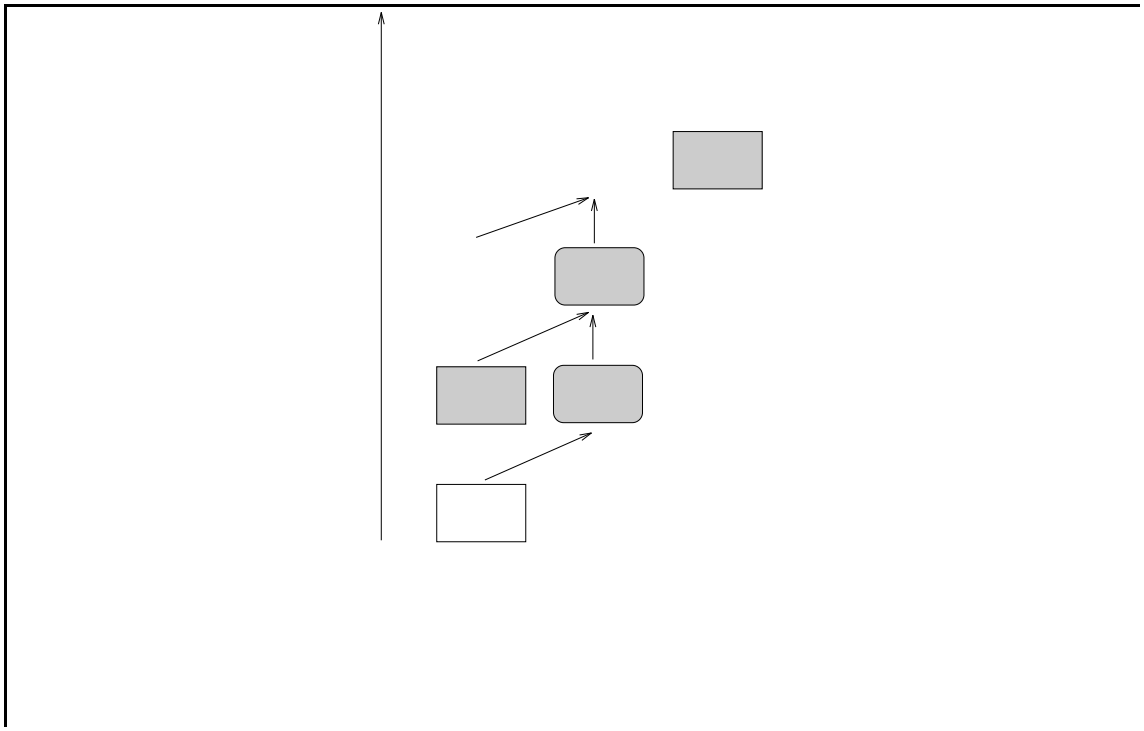
C C C C C CC
C C C C
C C C C
C C C C C ,
C C C C C
C C C C
C C C C C


There are bugs and s ips




"Multiply"
"Write number"
Tens
Units
"Next column"
"Done"
ALU
Hidden
Copy
Predicates
Context
x
y

Figure



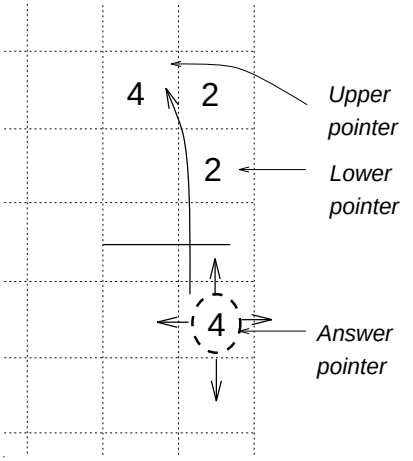


Figure C C

top_next_column (TNC)	store_mark (STR)
ump_answer_space (JAS)	zero_accumulator (ZAC)
ump_top_row (JTR)	next_answer_row (NAR)
left (LFT)	next_bottom_column (NBC)
right (RHT)	inc_answer_column (IAC)
up (UP_)	inc_top_column (ITC)
down (DWN)	add_start_position (SAD)
read_carry (RDC)	start_multiplication (SMU)

write_units (UNI)	add_mark_to_accumulator (ADD)
write_tens (TEN)	compute_product (MUL)
mark_zero (MKZ)	draw_rule (RUL)
mark_carry (MKC)	done (DON)

Tab e

add_start_position

start_multiplication

mark_carry

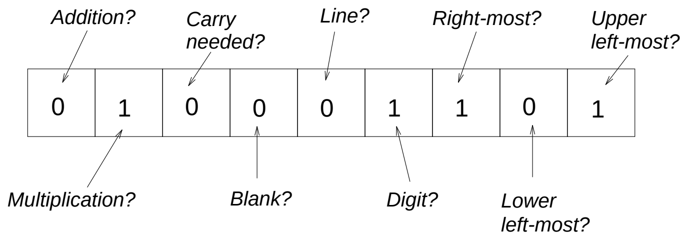
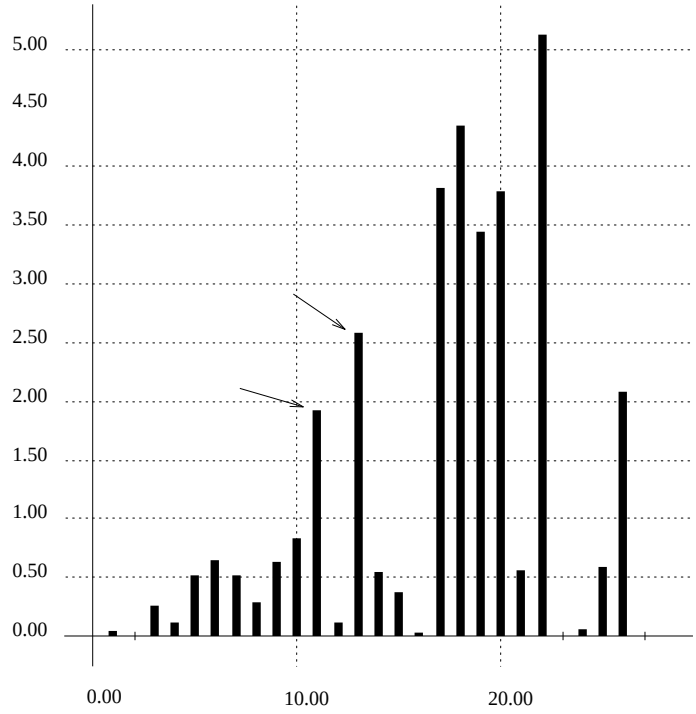


Figure C

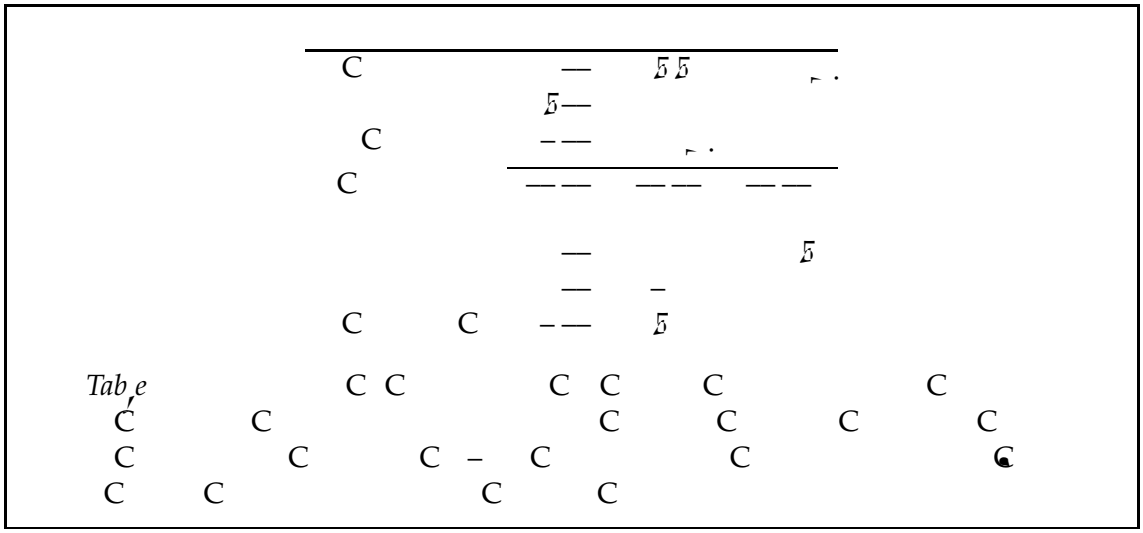
C C C C C C C C C
C C C C C C C
C `top_next_column` C C CC C C C
C C C C C C C
C C C C C
C C C C C C
C C C C C C C C
C C C C C C C C C
C C C C C C

C C C C C C

×	start_multiplication	C	C	-
×	store_mark			-
×	ump_top_row	C	C	-
×	compute_product			
×	ump_answer	C	C	
×				


5.00
4.50
4.00
3.50
3.00
2.50
2.00
1.50
1.00
0.50
0.00
0.00
10.00
20.00

C C C
C C C C
C C C
1 1 1
× 1 1



Correct behaviour

Star bugs

*P, ausib, e co bination*s

C C C C C C C C ,

C ,– , C –
C , C, ,C – —
C , C, C ,— Б Б Б .
C , C, , .

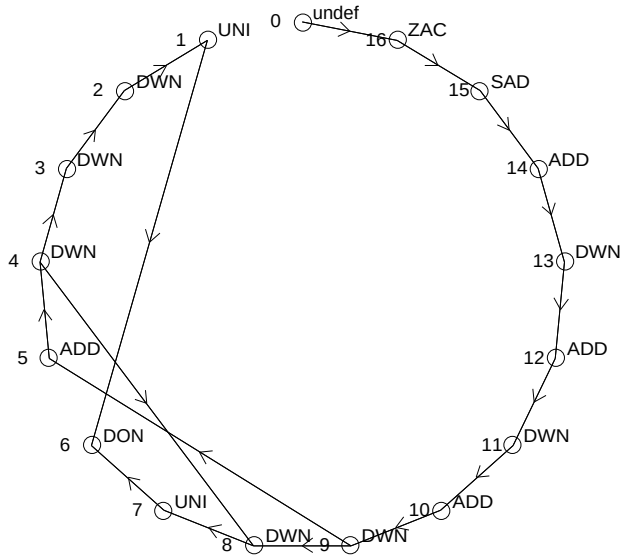


Figure C C

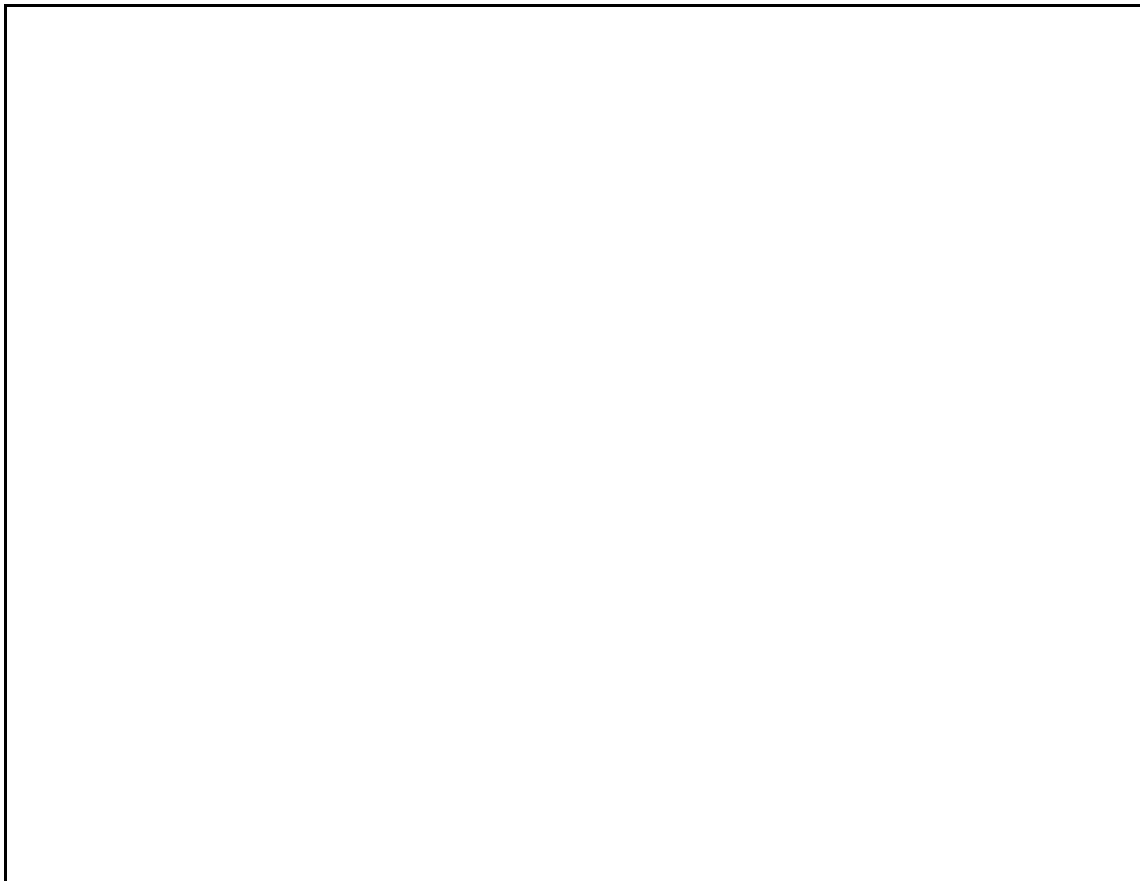
C C C --

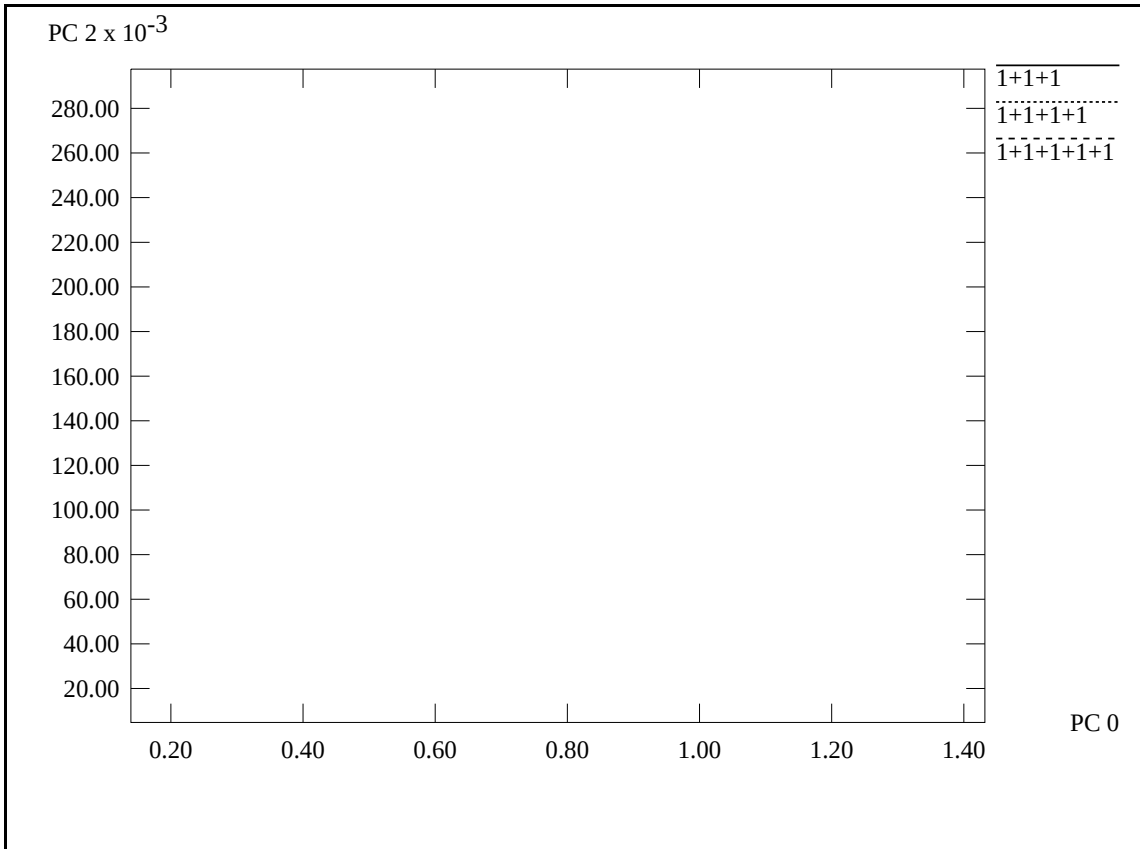
C C C

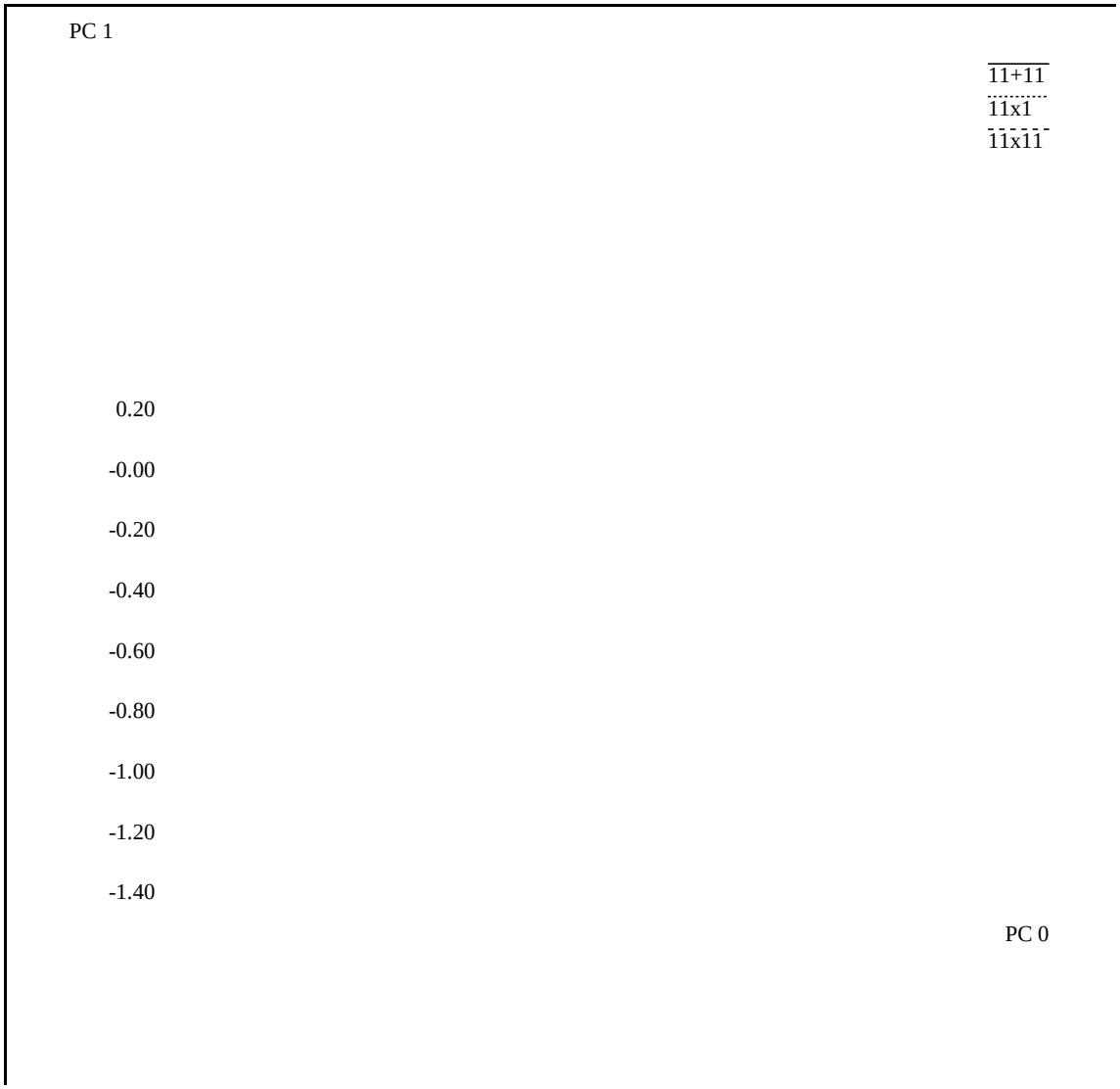
C DWN C C C

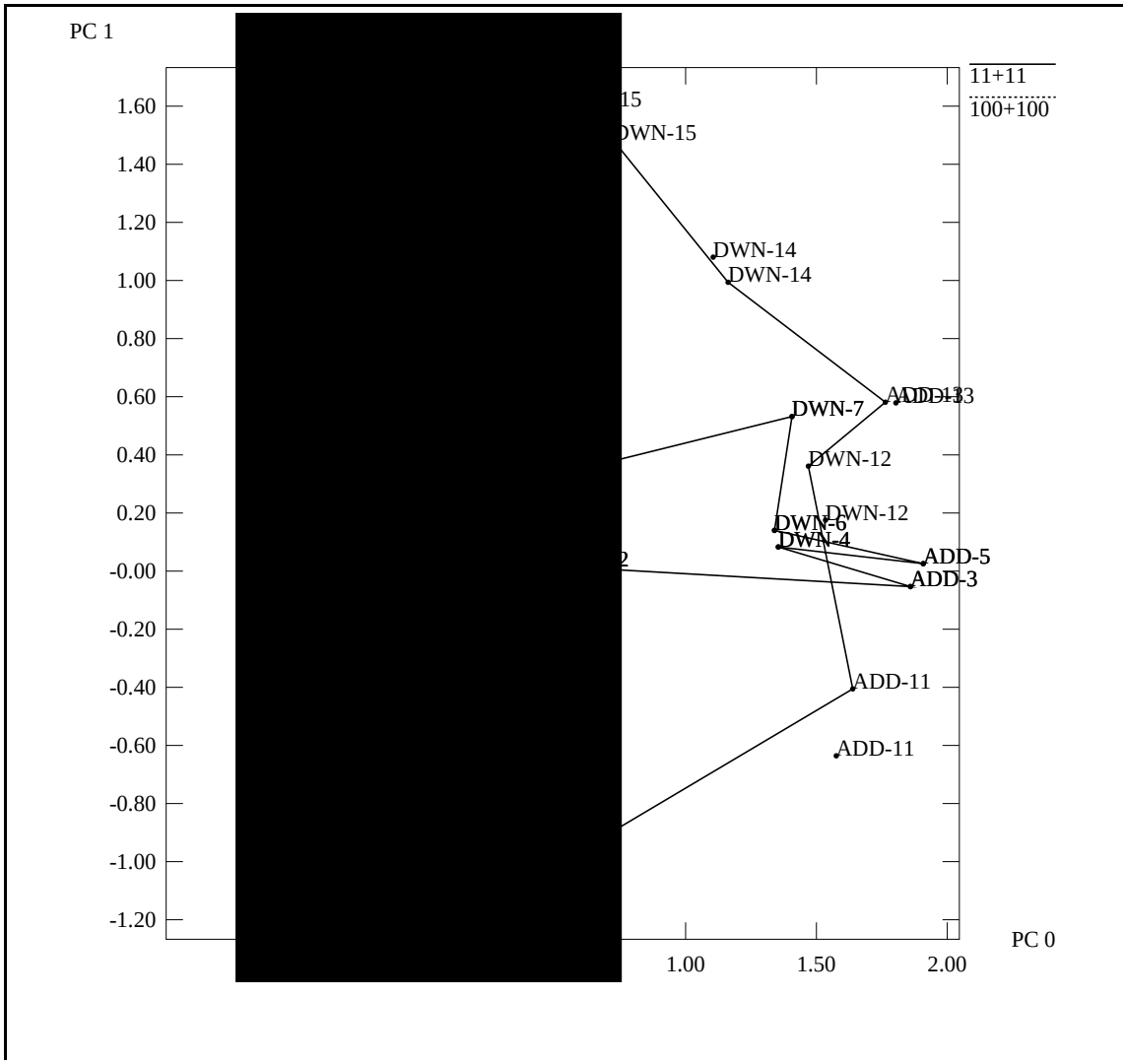
C C C

C C




PC 2 x 10-3
280.00
260.00
240.00
220.00
200.00
180.00
160.00
140.00
120.00
100.00
80.00
60.00
40.00
20.00
0.20
0.40
0.60
0.80
1.00
1.20
1.40
PC 0
1+1+1
1+1+1+1
1+1+1+1+1


PC 1
11+11
11x1
11x11
0.20
-0.00
-0.20
-0.40
-0.60
-0.80
-1.00
-1.20
-1.40
PC 0


PC 1
11+11
100+100
1.60
1.40
1.20
1.00
0.80
0.60
0.40
0.20
-0.00
-0.20
-0.40
-0.60
-0.80
-1.00
-1.20
DWN-14
DWN-14
DWN-7
DWN-12
DWN-12
DWN-6
DWN-4
ADD-5
ADD-3
ADD-11
ADD-11
1.00
1.50
2.00
PC 0

C C C C C C C C C
C C C C C
C C C

Bug igration as noise

$$\text{net}_i = \sum_j (a_j w_{ij} + h(\frac{w_{ij}}{30}))$$

$h(n)$ $\pm n$

–

×   ×

```
  1 1 1
× 1   1
-------
    1 1
```

–

```
  1 1 1          1 1 1
×     1        ×     1
-------        -------
  1 1 1            1 1
```

–

```
  1 1 1          1 1 1          1 1 1
×   1 1        ×     1        ×     1
-------        -------        -------
  1 1 1            2 1          2 1 1
                                2
```

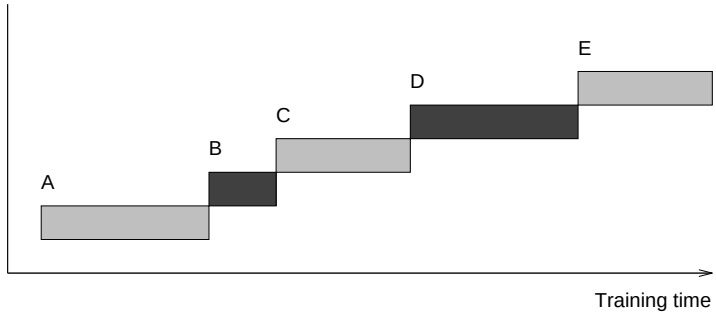


Figure C C C C C ,
C C C

5.7 Summary

mark_carry

Su ary

C

C C C C C C

C C C C C

C

6.1 Memory for arithmetic facts

C C C C

Bibliography

Connection Science

. C C C , C
C ,
C

. C C ,
C C *Instructiona, Science* . .

. C C C
C . C C C
C C C C C *Brain Deve,op ent and Cognition*
A Reader C

C . *Co puter Mode,s of Mind*

C C . C C C C
Cognitive Science .

C . C C C C
C *Cognitive Science* . 5

C . C C
C *Inte,,igent Tutoring Syste s* C C
. 5 *Diagnostic Studies in Arith etic* C C
C

. C C C
C C C *Cognitive Processes in Mathe atics*
– C C

. C C
C *Canadian Journa, of Psycho,ogy* 55

. C C C C
C C *Connection Science* C

C . C C C
C C *Neura, Connections*
Menta, Co putation

. CC C C
. CC C C C C C C

. 5 CC C *Mind Language* 5

. C C C
Advances in Arti cia, Inte,,igence C C

. *Mind Language* —

. *Microcognition Phi,osophy Cognitive Science and Para,,e, Distributed*
Processing

Proceedings of the Eight Annual Conference of the Cognitive Science Society

Information Thought and Content

C . . C C C C .
Advances in Neura, Infor ation Processing Syste s C C
C

C . . C C C
Arti cia, Inte,,igence –5

. *Co putation and Cognition Towards a Foundation for Cognitive Science*

. C C C C C C C C C
C C *Psycho,ogica, Review* –

. C . *The Psycho,ogy of Mathe atics for Instruction*
C

C C . C C
C C ., C C
C CC C C

C . C C C .
C . C *Proceedings of the Connectionist Mode,s Su er Schoo,* 5 5 C C

C C C C C *Machine Learning* 5 .

C C C C C C *Machine Learning Proceedings of the Eighth Internationa, or shop* – – . C C

, C C C C C *Proceedings of the Thirteenth Annua, Conference of the Cognitive Science Society* 5 5 – C

C C C C *Cognitive Processes in Mathe atics* –5 C C

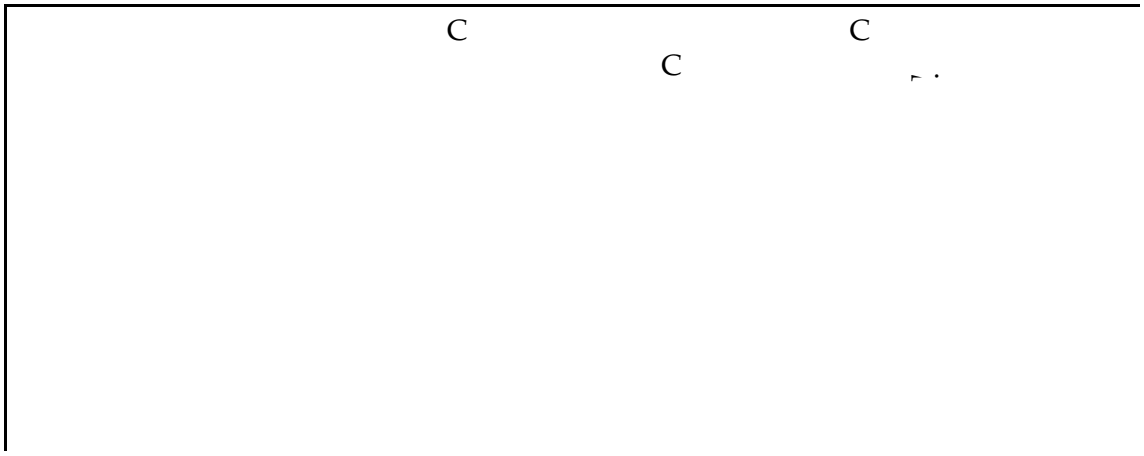
C C C C C *Journa, of Experi enta, Psycho,ogy Genera,*

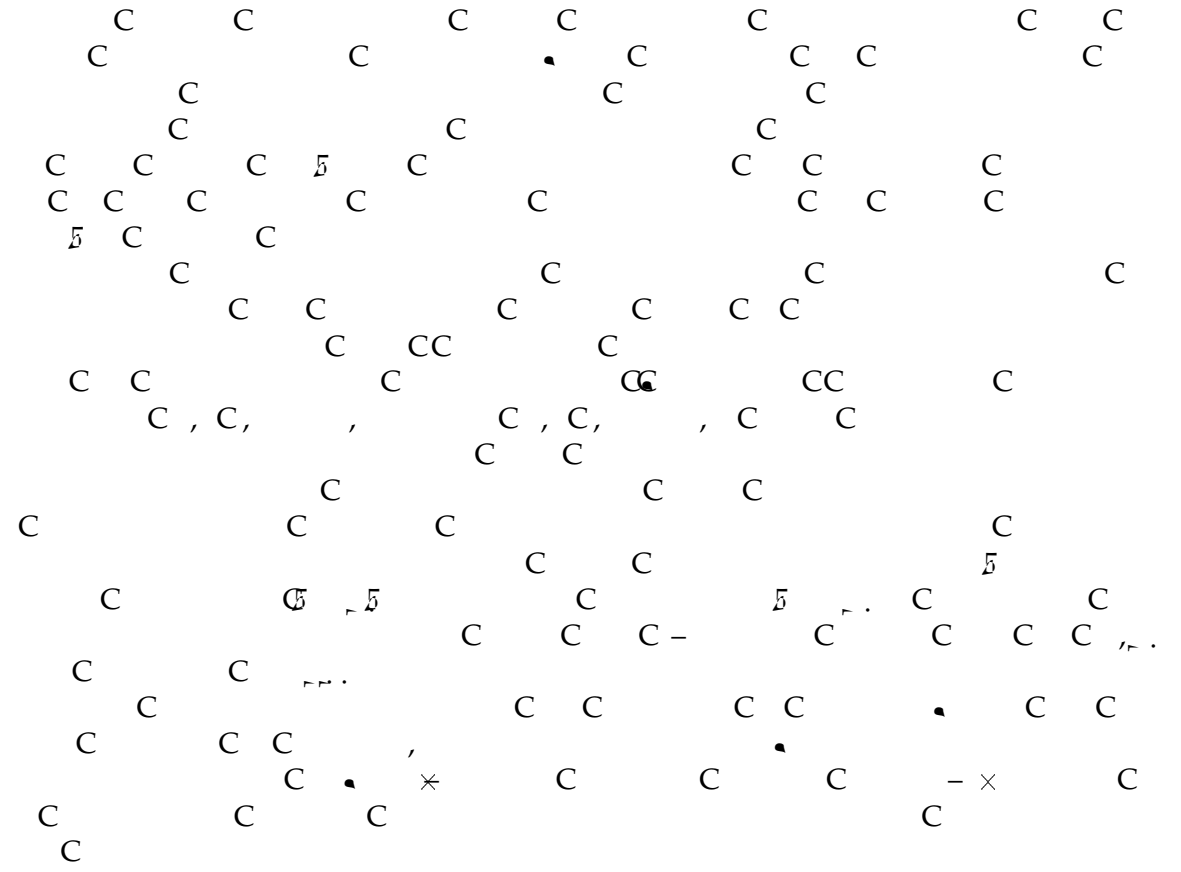
C C C C C C C *Origins of Cognitive S i,,s The Eighteenth Annua, Carnegie Sy posiu of Cognition* . .
C

C C
C *Inte,,igent Tutoring Syste s* .
C C

C C C C
C CC C C C C
C

C C C


C
C
C



Addition bugs

Added-imaginary-column.

5 .

```
  4 6
+   3
-----
  7 9
```

Carries-one-to-100s. C C C
C C

```
  5 0 5
+   7 4
-------
  6 7 9
```

C – 5

Carries-one-to-10s. C C C

```
  4 6
+   3
-----
  5 9
```

C

Carries-ten. C –

```
  2 5
+ 1 7
-----
1 3 2
```

Carries-two. C C

```
  2 7 1
+ 4 1 2
-------
  8 0
```

r870406(e)5

Column-skipped. C C C

$$\begin{array}{r} 3\ 7\ 5 \\ +\ 2\ 1\ 2 \\ \hline 5\quad 7 \end{array}$$

Does-not-raise-carry. C C C C C C

$$\begin{array}{r} 7\ 8 \\ +\ 7\ 1 \\ \hline {\scriptstyle 1}\ 4\ 9 \end{array}$$

Does-not-record-100s. C C C C C

$$\begin{array}{r} 5\ 0\ 5 \\ +\ \ 7\ 4 \\ \hline 0\ 7\ 9 \end{array}$$

$$\begin{array}{r} 4\ 7\ 6 \\ +\ \ 1\ 7 \\ \hline 9_1\ 3 \end{array}$$

C

Does-not-rename-copy-100s. C C C C C C C C C C

$$\begin{array}{r} 2\ 0\ 5 \\ +\ \ 8\ 6 \\ \hline 2\ 1\ 1 \end{array}$$

Does-not-rename-quits-100s. C C C C C C C

$$\begin{array}{r} 2\ 0\ 5 \\ +\ \ 8\ 6 \\ \hline 8\ 1\ 1 \end{array}$$

Does-not-rename-sum. C C C C

$$\begin{array}{r} 4\ 8 \\ +\ \ 3 \\ \hline 4\ 1\ 1 \end{array}$$

$$\begin{array}{r} 2\ 8 \\ \times\ 1\ 7 \\ \hline 1_1\ 9_5\ 6 \\ +\ 2\ 8\ 0 \\ \hline 3\ 1\ 7\ 6 \end{array}$$

C

Ignores-10s-column. C C

$$\begin{array}{r} 4\ 8 \\ +\ \ 3 \\ \hline 1\ 1 \end{array}$$

Ignores-first-column. C C C C

$$\begin{array}{r} 3\ 2\ 5 \\ +\ 2\ 7\ 1 \\ \hline 5\ 9 \end{array}$$

Left-alignment. C C

$$\begin{array}{r} 5\ 4 \\ +\ 3 \\ \hline 8\ 4 \end{array}$$

C			
•			
	C , C, ,C		
	C , C,	Б–	

C			
⋇	–		
,C ,C , C	Б		

Mu,tip,ication Bugs

0×N=0-carry-N. C C C

```
  2 0
×   3
-----
  9 0
```

Adds-carry-and-multiplicand. C
C × C

```
  5 3 6
×     8
-------
  5 7 8
```

Adds-carry-and-multiplier. C
C × − × C

```
  8 0 5
×     4
-------
3 2 6 0
```

Adds-carry-and-multiplier-when-zero. C
CC ×
× −

```
  5 0 7
×     2
-------
1 0 3 4
```

Adds-carry-to-multiplicands. C C
× ×

```
  5 3 6
×     8
-------
  9 7 8
```

Adds-carry-to-product. C C C C
× 5

```
   5 2
 × 1 3
 -----
   6 6
+5 2 0
 -----
   5 8 6
```

C

Adds-instead-of-multiplying. C C C C

```
   7 2 5
 ×     3
 -------
   7 2 8
```

C 5

Adds-multiplicand-to-answer. C C
×5

```
   7 6
 ×   3
 -----
  8 1 8
```

Adds-using-multiplication-pattern. C C

```
   3 2 0
 ×     4
 -------
   7 6 4
```

C

Always-carries.

```
 2 4 2 9
       2
 -------
 5 9 5 1 8
```

–

Always-carries-one. C C C C C

```
   5 1 4
 ×     7
 -------
 3 5 8 2 8
```

Answer-on-one-row. C C C C

```
     2 3
   × 4 8
 ---------
 9 1 3 1 8 2 4
```

C 5

Answers-left-to-right. C ×. CC

```
   7 1 2
```

Carries-wrong-digit. C C C C C C

$$\begin{array}{r} 7\ 2\ 4 \\ \times\quad\ 6 \\ \hline 4\ 8_{6}\ 1_{4}\ 2 \end{array}$$

Carries-wrong-number. C C C C C , C , C C , ,C

Carry-added-to-multiplicand. C C ,

$$\begin{array}{r} 3\ 2\ 7 \\ \times\quad\ 6 \\ \hline 3\ 6_{3}\ 6_{4}\ 2 \end{array}$$

Carry-added-to-tens. C C C

$$\begin{array}{r} 2\ 6 \\ \times\ 1\ 4 \\ \hline 2\ 8_{2}\ 4 \\ +\ 2\ 6\ 0 \\ \hline 5_{1}\ 4\ 4 \end{array}$$

Carry-not-raised. C C C C

$$\begin{array}{r} 4\ 2 \\ \times\ 4\ 1 \\ \hline 4\ 2 \\ +_{1}6\ 8\ 0 \\ \hline 7_{1}\ 2\ 2 \end{array}$$

Carry-once-always-carry. C

$$\begin{array}{r} 1\ 1\ 2 \\ \times\quad\ 7 \\ \hline 8_{1}\ 8_{1}\ 4 \end{array}$$

Copies-after-first-column. C C C C C C , –

Copies-multiplicand. C C C C C
C

$$\begin{array}{r} 200 \\ \times \quad 4 \\ \hline 200 \end{array}$$

Does-not-carry-in-partial-product. C C
C

$$\begin{array}{rrrrr} & & 9 & 2 & 7 \\ & \times & & 7 & 3 \\ \hline & 2 & 7 & 8 & 1 \\ 6 & 4 & 8 & 9 & 0 \\ \hline 6 & 6 & 5 & 7 & 1 \end{array}$$

Does-not-carry-to-10s. C C C C

C

$$\begin{array}{rrrr} & 2 & 1 & 6 \\ \times & & & 6 \\ \hline 1 & 2 & 6_3 & 6 \end{array}$$

Incorrect-number-of-annex-zeros. C C C CC C
C

$$\begin{array}{r} 4\ 5\ 6 \\ \times\ 2\ 5\ 1 \\ \hline 4\ 5\ 6 \\ 2\ 2\ 8\ 0\ 0 \\ 9\ 1\ 2\ 0\ 0 \end{array}$$

Last-digits-multiplied.

× × −

×

$$\begin{array}{r} 5\ 0\ 7 \\ \times\ \ 3\ 2 \\ \hline 1\ 5\ 1_1\ 4 \end{array}$$

C

Last-multiplication-skipped. C C C

$$\begin{array}{r} 3\ 2 \\ \times\ 4\ 1 \\ \hline 3\ 2 \\ +\ 8\ 0 \\ \hline 1\ 1\ 2 \end{array}$$

C

Multiplied-product-by-carry. C

Multiplies-last-multiplicand-and-writes-10. C C C C
×5 C
C

```
  3 0
×   6
1 0 1 8
```

C –

Multiplies-multiplicands. C C ,
× ×

```
  2 4
× 3 1
  8 4
```

C

Multiplies-partial-product. C C
, , C ,

```
  3 2
× 2 1
  3 2
6 4 0
7 2 0
```

C 5

Multiplies-using-addition-pattern. C

```
  5 2 4
× 7 3 1
3 5 6 4
```

C

× − × C C C

C C

$$\begin{array}{r} 1\ 4\ 4 \\ 2\ 5 \\ \hline 3_1\ 0_2\ 0 \end{array}$$

$$\begin{array}{r} 5\ 1\ 2 \\ \times\quad 2\ 5 \\ \hline 5\ 1\ 2 \end{array}$$

Skips-zero-multiplicand. C C C
C
C ×. × –

```
  8 0 9
×   5 2
-------
4 0 1 8
```

Spurious-zero-in-100s. C C C C C
C –

```
    9 0 5
  ×   4 6
-----------
5 4 0 3 0
3 6 0 2 0
```

Subtracts-partial-product. C
C C
C –

```
    5 3
  × 7 4
-------
  2 1 2
3 7 1 0
-------
3 5 0 2
```

Too-many-annex-zeros.